3109 Multivariable Analysis Notes

Based on the 2011 autumn lectures by Dr I Petridis

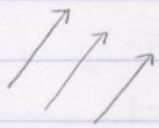
Introduction

$F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

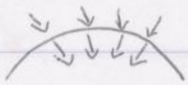
If $m=1$, F is called a scalar field

If $m>1$, F is called a vector field

Examples



Fluid flow



force field.

Differential forms, ω $\quad \displaystyle\int_{\partial m} \omega = \int_m d\omega$ Stokes theorem

What is ω?

What is $d\omega$?

What is a manifold m?

What is ∂M?

Motto: Differential forms are meant to be intergrated.

Newton $f'(x)$

Leibniz $\dfrac{df}{dx} = f'(x)$

not a quotient $df = f'(x)\,dx$

$F : \mathbb{R} \longrightarrow \mathbb{R}$ differentiable

$\displaystyle\int_a^b F'(x)\,dx = F(b) - F(a)$

~~1 - dimension $\mathbb{R}^1$~~

A differential form looks like

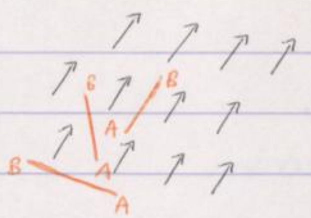
1-dimension $\mathbb{R}^1$

A differential form looks like $g(x)dx$

$[a,b]$ $\int_a^b g(x)dx$ is a real number.

2-dimension $\mathbb{R}^2$

Let $\vec{F}$ be a constant vector field



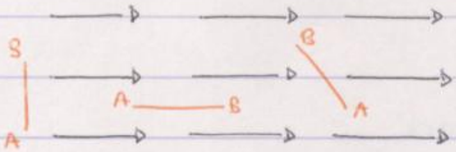
work is $= \vec{F} \cdot \vec{AB} = ax + by$

$\vec{F} = (a,b) = a\vec{i} + b\vec{j}$

$\vec{AB} = (x,y) = x\vec{i} + y\vec{j}$

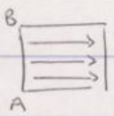
$$\int_A^B \vec{F}\, d\vec{r} = \int_A^B a\,dx + b\,dy$$

Fluid flow



$\vec{F} = v_1\vec{i} + v_2\vec{j}$

$\vec{AB} = x\vec{i} + y\vec{j}$

Area of rectangle 

Area is determinant $\begin{vmatrix} v_1 & v_2 \\ x & y \end{vmatrix} = v_1 y - v_2 x$

Green's Theorem (1828)

$\subseteq \mathbb{R}$

$$\int_\gamma f\,dx + g\,dy = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

In $\mathbb{R}^3$

$f(x,y,z)$ 0-form to be intergrated over summed up 0-chain, which is a collection of points.

$f(x,y,z)\, dx \wedge dy \wedge dz$ 3-form to be intergrated over solids

$$\vec{F} = (f, g, h)$$

1-form	2-form
$f\,dx + g\,dy + h\,dz$	$f\,dy \wedge dz + g\,dz \wedge dx + h\,dx \wedge dy$
intergrated over a curve	to be intergrated over a surface.

Operators.

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$
gradient

$\omega = f$ 0-form

$$d\omega = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$\vec{F} = (f, g, h)$

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f & g & h \end{vmatrix}$$

$$= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\vec{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\vec{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\vec{k}$$

$\omega = f\,dx + g\,dy + h\,dz$

$$d\omega = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)dy \wedge dz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)dz \wedge dx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dx \wedge dy$$

Divergence $\nabla . \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$

$\omega = f dy \wedge dz + g dz \wedge dx + h dx \wedge dy$

$d\omega = \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \right) dx \wedge dy \wedge dz$

$f \Rightarrow \nabla f \Rightarrow \text{curl}(\nabla f) = 0$

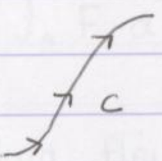
potential — conservative v.f — $d(d\omega) = 0$

$\omega = f$ 0-form — $d\omega$ 1-form — 2-form

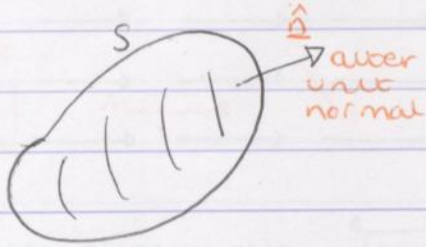
$\vec{F} \Rightarrow \text{curl}(\vec{F}) \Rightarrow d\omega(\text{curl } \vec{F}) = 0$

ω 1-form — $d\omega$ 2-form — $d(d\omega) = 0$.

Line Intergral



$\int_C \vec{F} . d\vec{r}$ $\quad \int_C f dx + g dy + h dz.$

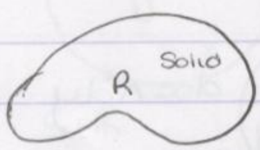


$\int_S \vec{F} \cdot \vec{n} \, d\sigma$

$\int_S f dy \wedge dz + g dz \wedge dx + h dx \wedge dy$

f-function $\int_S \frac{\partial f}{\partial n} d\sigma$ $\quad \int_S \frac{\partial f}{\partial x} dy \wedge dz + \frac{\partial f}{\partial y} dz \wedge dx + \frac{\partial f}{\partial z} dx \wedge dy$

Triple Intergral



$\iiint_R f dV$ $\quad \int_R f dx \wedge dy \wedge dz.$

If $\vec{F}$ is a potential ie $\vec{F} = \nabla \cdot f$

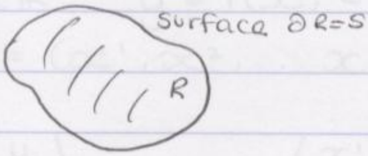
$$\int_A^B \nabla f \, d\bar{r} = f(B) - f(A) \qquad \int_c df = \int_{\partial c} f.$$

Work done by conservative field

Gauss Theorem (Divergence Theorem)

$$\int_R \nabla . \vec{F} \, dV = \int_{\partial R} \bar{F} . \bar{n} \, d\sigma$$

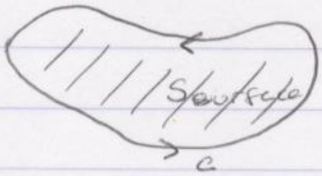
solid

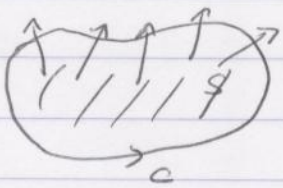


Flux of $\vec{F}$ through the boundary of R.

$$\int_R d\omega = \int_{\partial R.} \omega$$

Classical Stokes Theorem





unit normal $\hat{n}$

$$\int_c \bar{F} d\bar{r} = \int_S \text{curl} \, \bar{F} \times \bar{n} \, d\sigma$$

$\omega = f dx + g dy + h dz$

$$d\omega = \left(\frac{\partial h}{\partial y} - \frac{\partial y}{\partial z} \right) dy \wedge dz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) dz \wedge dx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy$$

Notation

$\mathbb{R}^n \ni x = (x^1, x^2, \ldots x^n) \quad x^i \in \mathbb{R}$

$\mathbb{R}^n$ is a vector space

Length - norm $|x| = \sqrt{(x^1)^2 + (x^2)^2 + \ldots + (x^n)^2}$

If $x, y \in \mathbb{R}^n \quad x \cdot y = x^1y^1 + x^2y^2 + \ldots + x^ny^n$

Standard basis $e_j = (0, 0, \ldots 0, \underset{j}{1}, 0, \ldots 0, 0) \quad j = 1, 2, \ldots n$

e_1 in $\mathbb{R}^2$ $(1,0)$ $\quad e_1$ in $\mathbb{R}^3$ $(1,0,0)$.

Properties of norm:

$|x| \geq 0$

$|x| = 0$ iff $x = 0$

$\underbrace{|\lambda \cdot x|}_{\text{norm}} = \underbrace{|\lambda|}_{\text{modulus}} \cdot \underbrace{|x|}_{\text{norm}} \quad x \in \mathbb{R}^n \quad \lambda \in \mathbb{R}$

(scalar multiplication on left, real number multip. on right).

Linear Transformation

$T : \mathbb{R}^n \to \mathbb{R}^m$

i) $T(\underbrace{x+y}_{\text{add}^n \text{ in } \mathbb{R}^n}) = \underbrace{T(x) + T(y)}_{\text{add}^n \text{ in } \mathbb{R}^m} \quad \forall x, y \in \mathbb{R}^n$

ii) $T(\underbrace{\lambda \cdot x}_{\text{scalar multi in } \mathbb{R}^n}) = \underbrace{\lambda \cdot T(x)}_{\text{scalar multi in } \mathbb{R}^m} \quad \forall \lambda \in \mathbb{R}$

Matrix representation of T w.r.t the standard basis of $\mathbb{R}^n$ & $\mathbb{R}^m$

$\mathbb{R}^m \ni T(\underbrace{e_j}_{\text{in } \mathbb{R}^n}) = \sum_{i=1}^{m} a_{ji} \underbrace{e_i}_{\text{in } \mathbb{R}^m}$

$[T]_\varepsilon^\varepsilon = A = (a_{ij})_{\substack{i=1\ldots m \\ j=1\ldots n}}$ matrix of size $m \times n$

$T: \mathbb{R}^n \to \mathbb{R}^m$

$S: \mathbb{R}^n \to \mathbb{R}^m$

$[T+S] = [T] + [S]$

λ scalar $[\lambda \cdot T] = \lambda \cdot [T]$

$U: \mathbb{R}^m \to \mathbb{R}^k$

$U \circ T: \mathbb{R}^n \to \mathbb{R}^k$

$[U \circ T]^{k \times m} = \underset{k \times m}{[U]}\underset{m \times n}{[T]}$

$T: \mathbb{R}^n \to \mathbb{R}^m$

$x \in \mathbb{R}^n \quad y = T(x) \in \mathbb{R}^m$

$x = (x^1, x^2, \dots x^n) \qquad y = (y^1, y^2, \dots y^m)$

$$\underset{m \times 1}{\begin{pmatrix} y_1 \\ y^2 \\ \vdots \\ y^m \end{pmatrix}} = \underset{m \times n}{[T]} \underset{n \times 1}{\begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix}}$$

Functions and Continuity.

$f : \mathbb{R}^n \to \mathbb{R}^m$ vector valued function

$f : A \to \mathbb{R}^m$ where $A \subseteq \mathbb{R}^n$

f has components which are scalar fields.

$f(x) = (f^1(x), f^2(x), f^3(x), \dots f^m(x))$ where $f^i : A \to \mathbb{R}$

$\pi^i : \mathbb{R}^m \to \mathbb{R}$

$\pi^i(x^1, x^2, \dots x^m) = x^i$ is a linear transformation. (prove)

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "f"] \arrow[dr, "f^i"'] & \mathbb{R}^m \arrow[d, "\pi^i"] \\ & \mathbb{R} \end{tikzcd}$$

Definition : Limit

$f : \mathbb{R}^n \to \mathbb{R}^m$

$\lim\limits_{\underset{\in \mathbb{R}^n}{x} \to \underset{\in \mathbb{R}^n}{a}} f(x) = \underset{\in \mathbb{R}^m}{b}$ means $\forall \varepsilon > 0 \ \exists \delta > 0$ st $0 < \underset{\text{norm in } \mathbb{R}^n}{|x-a|} < \delta \Rightarrow \underset{\text{norm in } \mathbb{R}^m}{|f(x)-b|} < \varepsilon$

Definition : Continuous

f is called continuous at a if $\lim\limits_{x \to a} f(x) = f(a)$

f is called continuous on the set A if it is continuous at a for all $a \in A$.

Combination Theorem

Assume $\lim\limits_{x \to a} f(x) = b$, $\lim\limits_{x \to a} g(x) = c$. Then

1. $\lim\limits_{x \to a} (f(x) \underset{\text{addition in } \mathbb{R}^m}{+} g(x)) = b \underset{\text{addition in } \mathbb{R}^m}{+} c$

2. $\lim_{x \to a} (\lambda \cdot f(x)) = \lambda \cdot b \quad \lambda \in \mathbb{R}$

Scalar multi in $\mathbb{R}^m$

3. $\lim_{x \to a} f(x) \cdot g(x) = b \cdot c$

dot product in $\mathbb{R}^m$

4. $\lim_{x \to a} |f(x)| = |b|$

norm in $\mathbb{R}^m$

Proof of 3. (others exercise)

$$f(x) \cdot g(x) - bc = f(x)g(x) - b \cdot g(x) + b \cdot g(x) - b \cdot c$$
$$= (f(x) - b) \cdot g(x) + b \cdot (g(x) - c)$$

$$|f(x) \cdot g(x) - b \cdot c| = |(f(x) - b) \cdot g(x) + b \cdot (g(x) - c)|$$

modulus in $\mathbb{R}^m$

Triangle
$$\leq |(f(x) - b) \cdot g(x)| + |b \cdot (g(x) - c)|$$

Cauchy Schwartz
$$\leq |f(x) - b| \cdot |g(x)| + |b| \cdot |g(x) - c|$$

$\to 0$ $\to 0$

Since $\lim_{x \to a} g(x) = c$ g is bounded in a neighbourhood of a. exercise

ie $\exists M \geq 0 \; \exists \delta > 0 \; |g(x)| \leq M$ for $|x - a| < \delta$

6 October

Remark

1. $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous iff $f^i : \mathbb{R}^n \to \mathbb{R}$ is continuous for $i = 1, 2, \dots, n$

2. Polynomial functions in n-variables $f(x^1, \dots, x^n)$ are continuous

3. Rational functions $R(x) = \dfrac{P(x)}{Q(x)}$ are continuous

where defined ie $Q(x) \neq 0$.

P, Q polynomials in n variables.

eg. $\frac{(x^1)^2 + 5x^2}{(x^1)^2 - (x^2)^2}$ $Q(x) = (x^1)^2 - (x^2)^2 = 0$ hyperbola in (x^1, x^2) plane.

Theorem: Linear transformations are continuous.

$T: \mathbb{R}^n \to \mathbb{R}^m$

Let $a \in \mathbb{R}^n$. Want to show $\lim_{h \to 0} T(a+h) = T(a)$, $h = (h^1, \ldots, h^n)$

$$\begin{aligned}
|T(a+h) - T(a)| &= |T(h)| \\
&= |T(h^1 e_1 + h^2 e_2 + \ldots + h^n e_n)| \\
&= |h^1 T(e_1) + h^2 T(e_2) + \ldots + h^n T(e_n)| \\
&\leq |h^1||T(e_1)| + |h^2||T(e_2)| + \ldots + |h^n||T(e_n)| \\
&\leq |h||T(e_1)| + |h||T(e_2)| + \ldots + |h||T(e_n)|
\end{aligned}$$

$|T(a+h) - T(a)| \leq M|h|$ where $M = \sum_{i=1}^{n} |T(e_i)|$

Given $\varepsilon > 0$, choose $\delta = \varepsilon / M$

Example

$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$ $(x,y) \neq (0,0)$

$\lim_{(x,y) \to (0,0)} f(x,y)$? Limit does not exist.

Assume $\lim_{(x,y) \to (0,0)} f(x,y) = l$.

$\forall \varepsilon > 0 \ \exists \delta > 0$ st $0 < |(x,y)| < \delta \Rightarrow |f(x,y) - l| < \varepsilon$

Plug $(x,0)$ into f $f(x,0) = \frac{x^2 - 0^2}{x^2 + 0^2} = \frac{x^2}{x^2} = 1$

Plug $(0,y)$ into f $f(0,y) = \frac{0^2 - y^2}{0^2 + y^2} = \frac{-y^2}{y^2} = -1$

If $|x| < \delta$, $|(x,0)| < \delta \Rightarrow |f(x,0) - l| < \varepsilon$ $|1 - l| < \varepsilon$

If $|y|<\delta$, $|(0,y)|<\delta \Rightarrow |f(0,y)|<\varepsilon$ $|-1-l|<\varepsilon$

Take $\varepsilon = 1/2$ contradiction.

$y = mx$, $m \in \mathbb{R}$

$$f(x,mx) = \frac{x^2 - m^2x^2}{x^2+m^2x^2} = \frac{(1-m^2)x^2}{(1+m^2)x^2} = \frac{1-m^2}{1+m^2}$$

Remark: checking along lines isn't always enough, check curves too.

Naive approach:

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{x\to 0} 1 = 1$$

$x \neq 0$ fix x, look at $\lim_{y\to 0} f(x,y) = \lim_{y\to 0} \frac{x^2-y^2}{x^2+y^2} = \frac{x^2}{x^2} = 1$

$$\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = \lim_{x\to 0}(-1) = -1$$

$y \neq 0$ fix y, look at $\lim_{x\to 0} \frac{x^2-y^2}{x^2+y^2} = \frac{-y^2}{y^2} = -1$

Example

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Show that f is continuous at $(0,0)$.

$|f(x,y)| < \varepsilon$? if $|(x,y)| < \delta$

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| \leq \frac{|x|\cdot|y|}{\sqrt{x^2+y^2}} \leq \frac{\sqrt{x^2+y^2}\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} = |(x,y)|$$

Given $\varepsilon > 0$ choose $\delta = \varepsilon$.

If the total degree of each monomial in numerator is greater than the total degree in denominator then limit should be 0.

Theorem:

If f is continuous at a and g is continuous at $f(a)$, then $g \circ f$ is continuous at a.

Partial Derivatives

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $a \in \mathbb{R}^n$

Definition:

$$D_i f(a) = \lim_{h \to 0} \frac{f(a^1, a^2, \ldots, a^{i-1}, a^i + h, a^{i+1}, \ldots, a^n) - f(a)}{h}.$$

If $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$
(a, b)

$\frac{\partial f}{\partial x}(a,b) = D_1 f(a,b)$ $\qquad$ $\frac{\partial f}{\partial y}(a,b) = D_2 f(a,b)$

In $\mathbb{R}^3$ we also use $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$

Example

$f(x,y) = x^y$

$\frac{\partial f}{\partial x} = yx^{y-1}$ $\qquad$ $\frac{\partial f}{\partial y} = x^y \ln(x)$.

Example

$$f(x,y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x}$$

$$= \lim_{x \to 0} \frac{\frac{x^2 - 0^2}{x^2 + 0^2} - 1}{x}$$

$$= \lim_{x \to 0} \frac{\frac{x^2}{x^2} - 1}{x} = \lim_{x \to 0} \frac{0}{x} = 0.$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y}$$

$$= \lim_{y\to 0} \frac{\frac{0^2 - y^2}{0^2 + y^2} - 1}{y}$$

$$= \lim_{y\to 0} \frac{-1-1}{y} = \lim_{y\to 0} \frac{-2}{y} = \pm\infty.$$

In 1-dimension $f: \mathbb{R} \to \mathbb{R}$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Try in higher dimensions.

$f: \mathbb{R}^n \to \mathbb{R}^m$

$\lim_{h\to 0} \cancel{\frac{f(a+h) - f(a)}{h}}$ $\quad a \in \mathbb{R}^n \quad h \in \mathbb{R}^n$

$f(a) \in \mathbb{R}^m, \; f(a+h) \in \mathbb{R}^m$

$$0 = \lim_{h\to 0} \left[\frac{f(a+h) - f(a)}{h} - f'(a) \right]$$

$$= \lim_{h\to 0} \left| \frac{f(a+h) - f(a) - h \cdot f'(a)}{h} \right|$$

$$= \lim_{h\to 0} \frac{|f(a+h) - f(a) - hf'(a)|}{|h|}$$

Derivative, Total Derivative

$f : \mathbb{R} \to \mathbb{R}$

$a \in \mathbb{R}$

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - hf'(a)}{h} = 0$$

Tangent line at a, $y = f(a) + f'(a)(x-a)$

Call $x - a = h$

$f(a) + h \cdot f'(a)$ Not a linear transformation

Look at map $h \xrightarrow{\lambda} hf'(a)$ $h \in \mathbb{R}$

This is a linear map

$h_1 + h_2 \to (h_1 + h_2) f'(a)$ $\lambda(h_1 + h_2)$

$= h_1 f'(a) + h_2 f'(a)$

$= \lambda(h_1) + \lambda(h_2)$.

$f : \mathbb{R}^n \to \mathbb{R}^m$ $\displaystyle \lim_{h \to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0$

Definition.

$f : \mathbb{R}^n \to \mathbb{R}^m$ (or $f : A \to \mathbb{R}^m$, A open in $\mathbb{R}^n$) is differentiable at a ($a \in A$) if we can find a linear transformation

$\lambda : \mathbb{R}^n \to \mathbb{R}^m$ st.

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0$$

The linear transformation λ is called the (total) derivative of f at a and denoted $Df(a)$ st $Df(a)(h) = \lambda(h)$

Example.

$f: \mathbb{R}^n \to \mathbb{R}^m$ which is constant $f(x) = K$
Is differentiable at $a \in \mathbb{R}^n$ with $Df(a) = 0$ which is the
0 linear transformation $0: \mathbb{R}^n \to \mathbb{R}^m$, $0(h) = 0 \in \mathbb{R}^m$.

$$\frac{|f(a+h) - f(a) - 0(h)|}{|h|} = \frac{|K - K - 0|}{|h|} = 0 \xrightarrow[h \to 0]{} 0.$$

Example

$f: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transform is differentiable at $a \in \mathbb{R}^n$
$Df(a) = f$.

$$\begin{aligned} f(a+h) - f(a) - Df(a)(h) &= f(a+h) - f(a) - f(h) \\ &= f(a+h-a-h) \\ &= f(0) \quad \text{(linear)} \\ &= 0 \end{aligned}$$

$f: \mathbb{R} \to \mathbb{R}$
$f(x) = mx$
$f'(a) = m$
$Df(a) = f$ $\qquad$ $Df(a)$ is a linear transform $\mathbb{R} \to \mathbb{R}$.
$Df(a)(h) = f(h) = mh$
$f'(a) = m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

$$\frac{f(a+h) - f(a) - hf'(a)}{h} \to 0$$

Theorem

if f is differentiable at a, then there exists a unique $\lambda: \mathbb{R}^n \to \mathbb{R}^m$ linear transformation such that

$$\lim_{h\to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0$$

Proof: Suppose $\mu: \mathbb{R}^n \to \mathbb{R}^m$ is another linear transform st

$$\lim_{h\to 0} \frac{|f(a+h) - f(a) - \mu(h)|}{|h|} = 0$$

Want $\lambda = \mu$ ie $\forall h \in \mathbb{R}^n$ $\lambda(h) = \mu(h)$.

$$\frac{|\lambda(h) - \mu(h)|}{|h|} = \frac{|\lambda(h) + f(a) - f(a+h) + f(a+h) - f(a) - \mu(h)|}{|h|}$$

$$\leq \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} + \frac{|f(a+h) - f(a) - \mu(h)|}{|h|}$$

$$\xrightarrow[h\to 0]{} 0 + 0 = 0$$

$$\lim_{h\to 0} \frac{|\lambda(h) - \mu(h)|}{|h|} = 0 \quad *$$

Let $h = 0$ $\lambda(h) = 0 = \mu(h)$ since μ, λ linear.

Shrink! (fix $h \in \mathbb{R}^n$, $h \neq 0$)

Let $t \in \mathbb{R}$, $th \in \mathbb{R}^n$, $t \to 0 \in \mathbb{R}$

$th \to 0 \in \mathbb{R}^n$

Plug th in $*$

$$0 = \lim_{t\to 0} \frac{|\lambda(th) - \mu(th)|}{|th|} = \lim_{t\to 0} \frac{|t\lambda(h) - t\mu(h)|}{|t||h|}$$

$$= \lim_{t\to 0} \frac{\cancel{|t|}\,|\lambda(h) - \mu(h)|}{\cancel{|t|}\,|h|}$$

$$= \frac{|\lambda(h) - \mu(h)|}{|h|} = 0$$

$\Rightarrow |\lambda(h) - \mu(h)| = 0 \Rightarrow \lambda(h) = \mu(h)$

Definition.

$f : \mathbb{R}^n \to \mathbb{R}^m \quad a \in \mathbb{R}^n$

f is differentiable at a

$Df(a) : \mathbb{R}^n \to \mathbb{R}^m$ is linear

Its matrix representation is denoted by $f'(a) \in M_{m \times n}$ and is called the Jacobian of f at a.

Example.

$f : \mathbb{R}^2 \to \mathbb{R}^2$

$x, y \in \mathbb{R}$

$f(x,y) = (x^2 y, x+5)$

Show that $Df(1,2)(h^1, h^2) = (4h^1 + h^2, h^1)$.

$f((1,2) + (h^1,h^2)) - f(1,2) - Df(1,2)(h^1,h^2)$

$= f(1+h^1, 2+h^2) - f(1,2) - (4h^1+h^2, h^1)$

$= ((1+h^1)^2(2+h^2) ; 1+h^1+5) - (2,6) - (4h^1+h^2, h^1)$

$= (2+h^2+(h^1)^2 2 + (h^1)^2 h^2 + 2h^1h^2 + 4h^1, 6+h^1) - (2,6) - (4h^1+h^2, h^1)$

$= (2 + h + 2(h^1)^2 + (h^1)^2 h^2 + 2h^1h^2 + 4h^1 - 2 - 4h^1 - h^2, 6+$

$= (2(h^1)^2 + (h^1)^2 h^2 + 2h^1 h^2, 0)$

$| \quad | = |2(h^1)^2 + (h^1)^2 h^2 + 2h^1h^2|$

$\leq 2|h|^2 + |h|^2|h| + 2|h||h| = 4|h|^2 + |h|^3$

$$\frac{|f((1,2)+(h^1,h^2)) - f(1,2) - Df(1,2)(h^1,h^2)|}{|h|} \leq \frac{4|h|^2 + |h|^3}{|h|}$$

$$= 4|h| + |h|^2 \xrightarrow[|h| \to 0]{} 0$$

$f'(a)$ is a matrix representation of $Df(a)$

$$= \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \underset{m\times n}{f'(a)} \begin{pmatrix} h^1 \\ h^2 \\ \vdots \\ h^n \end{pmatrix}$$

$$= Df(a)(h).$$

In 1402

$$f'(a) = \begin{pmatrix} D_1 f^1(a) & D_2 f^1(a) & \cdots & D_n f^1(a) \\ D_1 f^2(a) & D_2 f^2(a) & \cdots & D_n f^2(a) \\ \vdots & & & \\ D_1 f^m(a) & D_2 f^m(a) & \cdots & D_n f^m(a) \end{pmatrix}$$

$f(x,y) = (x^2y,\ x+5)$

$$\frac{\partial f^1}{\partial x} = 2xy \qquad \frac{\partial f^1}{\partial y} = x^2 \qquad\qquad f'(1,2) = \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}$$

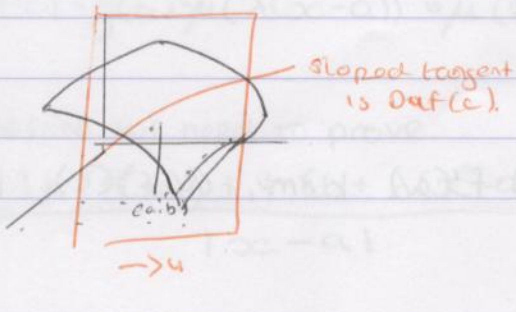
$$\frac{\partial f^2}{\partial x} = 2y \qquad \frac{\partial f^2}{\partial y} = 0$$

$$f'(1,2)\begin{pmatrix} h^1 \\ h^2 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} h^1 \\ h^2 \end{pmatrix} = \begin{pmatrix} 4h^1 + h^2 \\ h^1 \end{pmatrix}$$

Definition:

Let $u \neq 0$, $u \in \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}^m$.
The directional derivative of f at a in the direction u is given by

$$D_u f(a) = \lim_{h\to 0} \frac{f(a+hu) - f(a)}{h} \qquad h \in \mathbb{R}.$$



Remark: Having directional derivatives in all directions $u \neq 0$ is not enough to guarentee $Df(a)$ exists.

Theorem

If f is differentiable at a then f is continuous at a.

Proof: $\lim_{h \to 0} |f(a+h) - f(a)| = \lim_{h \to 0} |f(a+h) - f(a) - Df(a)(h) + Df(a)(h)|$

$$\leq \lim_{h \to 0} \underbrace{\frac{|f(a+h) - f(a) - Df(a)(h)|}{|h|}}_{\to 0} |h| + \underbrace{|Df(a)(h)|}_{\to 0}.$$

Since $Df(a)$ is linear transformation $Df(a)$ is continuous

$\lim_{h \to 0} |Df(a)(h)| = |D(f(a)(0))| = |0| = 0$

Chain Rule:

$f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a

$g: \mathbb{R}^m \to \mathbb{R}^k$ is differentiable at $f(a)$

Then $g \circ f: \mathbb{R}^n \to \mathbb{R}^k$ is differenticble at a.

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "Df(a)\ \text{linear}"] \arrow[dr, "D(g\circ f)(a)\ \text{linear}"'] & \mathbb{R}^m \arrow[d, "Dg(f(a))\ \text{linear}"] \\
& \mathbb{R}^k
\end{tikzcd}$$

$$D(g \circ f)(a) = Dg(f(a)) \overset{\text{composition}}{\circ} Df(a)$$

$$(g \circ f)'(a) = g'(f(a)) \overset{\text{matrix multi}}{\cdot} f'(a)$$

$f'(a)$ is $m \times n$, $g'(f(a))$ is $k \times m$. $g'(f(a)) \cdot f'(a)$ $k \times m$, $(g \circ f)(a)$ $k \times m$.

Remark: $\lim_{h \to 0} \frac{|f(a+h) - f(a) - Df(a)(h)|}{|h|} = 0$

set $a+h = x$, $h = x-a$

Set $\varphi(x) = f(x) - f(a) - Df(a)(x-a)$

Then f is differentiable at if we show

$\lim_{x \to a} \frac{|\varphi(x)|}{|x-a|} = 0.$

Proof (chain rule).

call $Df(a) = \lambda$, $Dg(f(a)) = \mu$

linear transf.

$f(a) = b \in \mathbb{R}^m$

By the remark, f differentiable at a means

$f(x) - f(a) - \lambda(x-a) = \varphi(x)$ with $\lim_{x \to a} \frac{|\varphi(x)|}{|x-a|} = 0.$

Similarly set $\psi(y) = g(y) - g(b) - \mu(y-b)$

st g differentiable at b means $\lim_{y \to b} \frac{|\psi(y)|}{|y-b|} = 0.$

We need to show that

$$\lim_{x \to a} \frac{|g(f(x)) - g(f(a)) - (\mu \circ \lambda)(x-a)|}{|x-a|} = 0$$

$$\begin{aligned} g(f(x)) - g(f(a)) &= g(f(a) + \lambda(x-a) + \varphi(x)) - g(b) \\ &= g(b + \lambda(x-a) + \varphi(x)) - g(b) \\ &= \mu(\lambda(x-a) + \varphi(x)) + \psi(b + \lambda(x-a) + \varphi(x)). \\ &\overset{\mu \text{ linear}}{=} \mu(\lambda(x-a)) + \mu(\varphi(x)) + \psi(b + \lambda(x-a) + \varphi(x)) \end{aligned}$$

$$g(f(x)) - g(b) - \mu(\lambda(x-a)) = \mu(\varphi(x)) + \psi(b + \lambda(x-a) + \varphi(x)).$$

Therefore we need to prove

$$\lim_{x \to a} \frac{|\mu(\varphi(x)) + \psi(b + \lambda(x-a) + \varphi(x))|}{|x-a|} = 0$$

Suffices to show $\lim_{x \to a} \frac{|\mu(\varphi(x))|}{|x-a|} = 0$

and $\lim_{x \to a} \frac{|\psi(b+\lambda(x-a)+\varphi(x))|}{|x-a|} = 0$ by triangle inequality.

μ is a linear transform

$\exists_M \geq 0$ st $|\mu(h)| \leq M|h|$

$$\frac{|\mu(\varphi(x))|}{|x-a|} \leq \frac{M|\varphi(x)|}{|x-a|} \xrightarrow[x \to a]{} 0.$$

set $y = b+\lambda(x-a)+\varphi(x)$

$$\frac{|\psi(y)|}{|x-a|} = \underbrace{\frac{|\psi(y)|}{|y-b|}}_{\substack{\downarrow \\ 0 \\ \text{as } y \to b.}} \underbrace{\frac{|y-b|}{|x-a|}}_{\text{is it bounded.}}$$

$$\frac{|y-b|}{|x-a|} = \frac{|\lambda(x-a)+\varphi(x)|}{|x-a|}$$

$$\leq \frac{|\lambda(x-a)|+|\varphi(x)|}{|x-a|}$$

$$= \frac{|\lambda(x-a)|}{|x-a|} + \frac{|\varphi(x)|}{|x-a|} \longrightarrow 0.$$

$$\leq \frac{M_1|x-a|}{|x-a|}$$

Jacobians $\quad \underset{k \times n}{(g \circ f)'(a)} = \underset{k \times m}{g'(b)} \overset{\text{matrix mult}}{\cdot} \underset{m \times n.}{f'(a)}$

Theorem:

i) Define $s: \mathbb{R}^2 \to \mathbb{R}$

$s(x,y) = x+y$

Then s is differentiable $Ds = s$

ii) $p: \mathbb{R}^2 \to \mathbb{R}$

$p(x,y) = x \cdot y$

Then p is differentiable and $Dp(a,b): \mathbb{R}^2 \to \mathbb{R}$ is linear

$Dp(a,b)(h,k) = ak + bh$

$p'(a,b) = (b,a)$.

Proof i) s is linear $s((x,y)+(x',y')) = s(x,y) + s(x',y')$

and $s(\lambda(x,y)) = \lambda s(x,y)$

$\Rightarrow Ds = s$.

$$\begin{aligned} s((x,y)+(x',y')) &= s(x+x', y+y') \\ &= x+x'+y+y' \\ &= (x+y)+(x'+y') = s(x,y) + s(x',y'). \end{aligned}$$

(finish).

Proof ii). Use definition of derivative

$$\begin{aligned} &p((a,b)+(h,k)) - p(a,b) - Dp(a,b)(h,k) \\ &= p(a+h, b+k) - p(a,b) - (ak+bh) \\ &= (a+h)(b+k) - a \cdot b - (ak+bh) \\ &= ab + hb + ak + hk - ab - ak - bh \\ &= hk \end{aligned}$$

$$\frac{|p((a,b)+(h,k)) - p(a,b) - Dp(a,b)(h,k)|}{|(h,k)|} = \frac{|hk|}{\sqrt{h^2+k^2}}$$

$$\leq \frac{\sqrt{h^2+k^2}\sqrt{h^2+k^2}}{\sqrt{h^2+k^2}}$$

$$= \sqrt{h^2+k^2} \to 0$$

as $(h,k) \to (0,0)$

Remark:

1 Let $T: \mathbb{R}^n \to \mathbb{R}^m$ &

To check it was linear we listed two properties

$T(x+y) = T(x) + T(y) \qquad x, y \in \mathbb{R}^n$

$T(\lambda x) = \lambda T(x) \qquad \lambda \in \mathbb{R}.$

We can also check instead

$T(\lambda x + y) = \lambda T(x) + T(y)$

2 Let $g^i: \mathbb{R}^n \to \mathbb{R}$ be a linear map

Such a map is called a linear functional

The set of linear functionals from $\mathbb{R}^n$ to $\mathbb{R}$ is called the dual space of $\mathbb{R}^n$ {notation $(\mathbb{R}^n)^*$

Now let $g^1, g^2, \ldots g^m$ be linear functionals, $g^i: \mathbb{R}^n \to \mathbb{R}$.

Then I can combine them to get a map $g: \mathbb{R}^n \to \mathbb{R}^m$ by

$g(x) = (g^1(x), g^2(x), \ldots, g^m(x)).$

$g: \mathbb{R}^n \to \mathbb{R}^m$ is linear. $x, y \in \mathbb{R}^n \; \lambda \in \mathbb{R}.$

$g(\lambda x + y) = \lambda g(x) + g(y)$

$$\begin{aligned}
g(\lambda x + y) &= (g^1(\lambda x + y), g^2(\lambda x+y), \ldots g^m(\lambda x + y)) \\
&= (\lambda g^1(x) + g^1(y), \lambda g^2(x) + g^2(y), \ldots, \lambda g^m(x) + g^m(y)) \\
&= (\lambda g^1(x), \lambda g^2(x) \ldots \lambda g^m(x)) + (g^1(y), g^2(y), \ldots, g^m(y)). \\
&= \lambda(g^1(x), \ldots, g^m(x)) + (g^1(y), \ldots, g^m(y)). \\
&= \lambda g(x) + g(y)
\end{aligned}$$

Let $[g^i]$ be the matrix repr of g^i $(1 \times n)$

$[g^i] = (g_1^i, g_2^i, \ldots, g_n^i)$

$$m \times n \quad [g] = \begin{pmatrix} g_1^1 & g_2^1 & \cdots & g_n^1 \\ g_1^2 & g_2^2 & & g_n^2 \\ \vdots & & & \\ g_1^m & g_2^m & & g_n^m \end{pmatrix}$$

Theorem :

$f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a if and only if f^i are differentiable at a, $i = 1, \dots, m$
and $Df(a) = (Df^1(a), Df^2(a), \dots, Df^m(a))$.

Proof : $\Rightarrow$ Assume f is differentiable at a
$f^i = \pi^i \circ f$ $\quad \pi^i(x^1, \dots, x^m) = x^i$ $\quad \pi^i$ linear $\quad D\pi^i = \pi^i$
Chain rule $\Rightarrow f^i$ is differentiable
$Df^i(a) = D\pi^i(f(a)) \circ Df(a)$.
$Df^i(a) = \pi^i \circ Df(a)$
This is the equation $Df(a) = (Df^1(a), \dots, Df^m(a))$.

$\Leftarrow$ Assume all f^i are differentiable at a $\quad i = 1, 2 \dots m$

$$\begin{aligned}
&f(a+h) - f(a) - (Df^1(a)(h), Df^2(a)(h), \dots, Df^m(a)(h)) \\
&= (f^1(a+h), f^2(a+h), \dots, f^m(a+h)) - (f^1(a), f^2(a), \dots, f^m(a)) \\
&\qquad - (Df^1(a)(h), Df^2(a)(h), \dots, Df^m(a)(h)) \\
&= (f^1(a+h) - f^1(a) - Df^1(a)(h), \dots, f^m(a+h) - f^m(a) - Df^m(a)(h))
\end{aligned}$$

$$\frac{|f(a+h) - f(a) - Df(a)(h)|}{|h|} \leq \underbrace{\frac{|f^1(a+h) - f^1(a) - Df^1(a)(h)|}{|h|}}_{\downarrow\, 0} + \dots + \underbrace{\frac{|f^m(a+h) - f^m(a) - Df^m(a)(h)|}{|h|}}_{\downarrow\, 0}$$

as $h \to 0$.

Remark :

If $T, S : \mathbb{R}^n \to \mathbb{R}^m$ are linear
$(T+S)(x) = T(x) + S(x)$ is linear $\quad T+S : \mathbb{R}^n \to \mathbb{R}^m$
If $\lambda \in \mathbb{R}$ define $(\lambda T)(x) = \lambda T(x)$
$\lambda T : \mathbb{R}^n \to \mathbb{R}^m$ is also linear.

~~Claim:~~

Corollary:

$f, g : \mathbb{R}^n \to \mathbb{R}$ differentiable at a

1 $D(f+g)(a) = Df(a) + Dg(a)$

2 Product Rule

$D(f \cdot g)(a) \neq \cancel{Df(a) \cdot g(a) + Dg(a) \cdot f(a)}$ never multiply scalar on right.

$= g(a) \cdot Df(a) + f(a) \cdot Dg(a)$

3 Quotient Rule

If $g(a) \neq 0$ $\quad D\left(\frac{f}{g}\right)(a) = \frac{1}{g(a)^2}\left(g(a)Df(a) - f(a) \cdot Dg(a)\right)$

Proof 1). $f, g : \mathbb{R}^n \to \mathbb{R}$ sum up.

$s : (f(x), g(x)) \longmapsto f(x) + g(x)$

$x \longrightarrow$

$\mathbb{R}^n \xrightarrow{(f,g)} \mathbb{R}^2 \xrightarrow{s} \mathbb{R}$

$f + g = s \circ (f, g)$

$D(f+g)(a) \overset{\text{chain rule}}{=} Ds(f(a), g(a)) \circ D(f, g)(a)$

$\overset{s \text{ linear}}{=} s \circ (Df(a), Dg(a))$

$= Df(a) + Dg(a)$

Proof 2). $\mathbb{R}^n \xrightarrow[(f,g)]{} \mathbb{R}^2 \xrightarrow[p]{} \mathbb{R}$

$fg = p \circ (f, g)$

$D(fg)(a) = Dp(f, g)(a) \circ D(f, g)(a)$

$= Dp(f(a), g(a)) \circ (Df(a), Dg(a))$

$h \in \mathbb{R}^2$ $\quad D(fg)(a) : \mathbb{R}^2 \to \mathbb{R}$

$D(fg)(a)(h) = Dp(f(a), g(a)) \circ (Df(a), Dg(a))(h)$

$= Dp(f(a), g(a))(Df(a)(h), Dg(a)(h))$

$= f(a) \cdot Dg(a)(h) + g(a) Df(a)(h)$

$f : \mathbb{R}^n \to \mathbb{R} \quad a \in \mathbb{R}^n$

$$D_i f(a) = \lim_{h \to 0} \frac{f(a^1, a^2, \ldots, a^{i-1}, a^i + h, a^{i+1}, \ldots, a^n) - f(a)}{h}$$

If $D_i f(a)$ exists for all a in, say some open set U, then we get a function $D_i : U \to \mathbb{R}$

$$x \to D_i f(x)$$

Then we can talk about the partial derivatives of $D_i f$

eg $D_j(D_i f)(x) = D_{i,j} f(x)$

If $D_j f(x)$ exists for all $x \in U$ this is a function of x and we can consider $D_i(D_j f)(x) = D_{j,i}(x)$

(In general $i \neq j$)

eg. $f(x,y) = x^3 y^5$

$D_1 f(x,y) = 3x^2 y^5 \qquad D_2 f(x,y) = 5x^3 y^4$

$D_{2,1} f(x,y) = 15x^2 y^4 \qquad D_{1,2} f(x,y) = 15x^2 y^4$

Theorem

If $D_{i,j} f$ and $D_{j,i} f$ are continuous on an open set containing a, then $D_{i,j} f(a) = D_{j,i} f(a)$

Proof: In the exercises of Hwk 5.

Theorem

$A \subseteq \mathbb{R}^n$, If the max or min of $f : A \to \mathbb{R}$ occur at a point a in the ~~maxim~~ interior of A and $D_i f(a)$ exists then,

$$D_i f(a) = 0.$$

Proof: Consider $h(x) = f(a^1, a^2, \ldots, a^{i-1}, x, a^{i+1}, \ldots, a^n)$

x is an open interval around a^i

Since f has a max or min at a, h has a max or min at a^i

$\frac{dh}{dx}(a^i) = D_i f(a)$

By Analysis II $\frac{dh}{dx}(a^i) = 0 \Rightarrow D_i f(a) = 0$. $\square$

Recall: $f: \mathbb{R}^n \to \mathbb{R}^m$ $a \in \mathbb{R}^n$

$Df(a): \mathbb{R}^n \to \mathbb{R}^m$ linear map, total derivative

Jacobian $f'(a) \in M_{m \times n}$ is matrix rep. of $Df(a)$ in standard bases.

Theorem

If $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a, then $D_j f^i(a)$ exists for all $i = 1, \dots m$, $j = 1, \dots n$ and the Jacobian matrix is $f'(a) = (D_j f^i(a))$

$$f'(a) = \begin{bmatrix} D_1 f^1(a) & D_2 f^1(a) & \cdots & D_n f^1(a) \\ D_1 f^2(a) & D_2 f^2(a) & \cdots & D_n f^2(a) \\ \vdots & \vdots & & \vdots \\ D_1 f^m(a) & D_2 f^m(a) & \cdots & D_n f^m(a) \end{bmatrix}$$

$f(x) = (f^1(x), f^2(x), \dots, f^m(x))$ $\quad f^i: \mathbb{R}^n \to \mathbb{R}$.

Proof: Case $m = 1$

$h: \mathbb{R} \to \mathbb{R}^n$ $\quad h(t) = (a^1, \dots, a^{i-1}, t, a^{i+1}, \dots a^n)$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "h"] \arrow[dr] & \mathbb{R}^n \arrow[d, "f"] \\ & \mathbb{R} \end{tikzcd}$$

$$\left.\frac{d}{dt}(f \circ h)\right|_{t = a^i} = D_i f(a)$$

h is differentiable, $h(t) = (a^1, a^2, \ldots, a^{i-1}, t, a^{i+1}, \ldots a^n)$
because its components are differentiable.

$h^1(t) = a^1$ const

$\vdots$

$h^{i-1}(t) = a^{i-1}$ const

$h^i(t) = t$ linear function

$\vdots$

$$Dh(t) = (Dh^1(t), Dh^2(t), \ldots, Dh^n(t))$$
$$= (0, 0 \ldots, \underset{i-1}{0}, \underset{i}{Id}, \underset{i+1}{0}, \ldots, 0).$$

$$h'(a^i) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i$$

Remark: If $g: \mathbb{R} \to \mathbb{R}$

$Dg(t_0): \mathbb{R} \to \mathbb{R}$ linear map

$g'(t_0)$ Jacobian 1×1 matrix $= \frac{dg}{dt}(t_0)$ $\mathbb{R}$ number

abuse of notation

Since f, h are differentiable the chain rule implies

$$\underset{1\times1}{(f \circ h)'(a^i)} = \underset{1\times n}{f'(h(a^i))} \cdot \underset{n\times1}{h'(a^i)} = \underset{1\times n}{f'(a)} \cdot \underset{n\times 1}{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}} \leftarrow i$$

matrix multip.

$$\frac{d}{dt}(f \circ h)(a^i) \qquad\qquad = (f'(a))^i \quad i \text{ entry of jacobians}$$

$$D_i f(a)$$

$$\Rightarrow f'(a) = (D_1 f(a), D_2 f(a), \ldots, D_n f(a))$$

Case $m>1$

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$f(x) = (f^1(x), \dots, f^m(x))$

Recall: $Df(a) = \underbrace{(\underbrace{Df^1(a)}_{\text{linear } \mathbb{R}^n \to \mathbb{R}}, \dots, Df^m(a))}_{\text{linear } \mathbb{R}^n \to \mathbb{R}^m}$

$$f'(a) = \left.\begin{pmatrix} (f^1)'(a) \\ (f^2)'(a) \\ \\ (f^m)'(a) \end{pmatrix}\right\}_{m\times n}$$

$(f^i)'(a)$ is a $1\times n$ matrix.

By case $m=1$ $\quad f'(a) = \begin{pmatrix} D_1 f^1(a) & D_2 f^1(a) & \dots & D_n f^1(a) \\ D_1 f^2(a) & D_2 f^2(a) & \dots & D_n f^2(a) \\ \vdots & \vdots & & \vdots \\ D_1 f^m(a) & D_2 f^m(a) & \dots & D_n f^m(a) \end{pmatrix}$

Example

$$G(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Fix a vector $u \in \mathbb{R}^2$, $u = (u^1, u^2) \neq (0,0)$ $\quad u^2 \neq 0$

Directional $D_u G(0,0) = \lim_{h\to 0} \dfrac{G((0,0)+hu) - G(0,0)}{h}$

$$= \lim_{h\to 0} \frac{G(hu^1, hu^2) - 0}{h}$$

$$= \lim_{h\to 0} \frac{(hu^1)^2 (hu^2)}{(hu^1)^4 + (hu^2)^2} \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{h^3 (u^1)^2 u^2}{h(h^4 (u^1)^4 + h^2 (u^2)^2)}$$

$$= \lim_{h\to 0} \frac{(u^1)^2 u^2}{h^2 (u^1)^4 + (u^2)^2}$$

$$= \frac{(u^1)^2 u^2}{(u^2)^2} = \frac{(u^1)^2}{u^2}$$

$u^2 = 0$

$$D_u G(0,0) = \lim_{h \to 0} \frac{G(hu^1, h \cdot 0)}{h} = \lim_{h \to 0} \frac{\frac{(hu^1)^2 \cdot 0}{(hu^1)^4 + 0}}{h} = 0.$$

$$\cancel{G(x, x^2) = \frac{x^2 x^2}{x^4 + x^4} = \frac{1}{2}}$$

$G(x,y)$ not continuous $\Rightarrow G(x,y)$ not differentiable.
(hwk 1)

Hwk 2: If f is differentiable at a, $f: \mathbb{R}^n \to \mathbb{R}$, then $D_u f(c)$ exists and $D_u f(a) = Df(a)(u)$.

Theorem (Handout 1)

$f: \mathbb{R}^n \to \mathbb{R}^m$. If $D_j f^i(x)$ exists $\forall x \in U$, U open, $a \in U$ and $j = 1, \dots, n$, $i = 1, \dots, m$. and are continuous at a
ie. $D_j f^i(x) \xrightarrow[x \to a]{} D_j f^i(a)$
the f is differentiable at a.

Proof: I can assume $m = 1$, $f: \mathbb{R}^n \to \mathbb{R}^m$, for simplicity $n = 2$.

$f(a^1 + h^1, a^2 + h^2) - f(a^1, a^2) - Df(a)(h^1, h^2)$
Candidate for $Df(a)$?
$f'(a) = (D_1 f(a), D_2 f(a))$

$$Df(a)(h^1, h^2) = f'(c) \begin{pmatrix} h^1 \\ h^2 \end{pmatrix}$$

$$= D_1 f(a) h^1 + D_2 f(a) h^2$$

$f(a^1 + h^1, a^2 + h^2) - f(a^1, a^2) - h^1 D_1 f(a) - h^2 D_2 f(a)$
$= f(a^1 + h^1, a^2 + h^2) - f(a^1 + h^1, a^2) + f(a^1 + h^1, a^2) - f(a^1, a^2)$
$\quad - h^1 D_1 f(a) - h^2 D_2 f(a) =$

Since $D_2 f$ exists and is continuous on an open set around a
$D_2 f$ " " " " segment $\Big|^{(a^1+h^1, a^2+h^2)}_{(a^1+h^1, a^2)}$.

Apply MVT in second variable
There exists a ξ^2 between a^2 and a^2+h^2 st
$f(a^1+h^1; a^2+h^2) - f(a^1+h^1, a^2) = D_2 f(a^1+h^1, \xi^2)h^2$
Apply MVT in first variable
There exists a ξ^1 between a^1 and a^1+h^1 st
$f(a+h^1; a^2) - f(a^1, a^2) = D_1 f(\xi^1, a^2) h^1$

$= D_2 f(a^1+h^1, \xi^2)h^2 + D_1 f(\xi^1, a^2)h^1 - D_1 f(a)h^1 - D_2 f(a)h^2$
$= h^2 [D_2 f(a^1+h^1, \xi^2) - D_2 f(a)] + h^1 [D_1 f(\xi^1, a^2) - D_1 f(a)]$

$|f(a+h) - f(a) - Df(a)(h)| = |h^1[D_1 f(c_1) - D_1 f(a)]$
$+ h^2 [D_2 f(c_2) - D_2 f(a)]|$

with $c_1 = (\xi^1, a^2)$
$c_2 = (a^1+h^1, \xi^2)$

$$\frac{|f(a+h) - f(a) - Df(a)(h)|}{|h|} = \frac{|h^1[D_1 f(c_1) - D_1 f(a)] + h^2[D_2 f(c_2) - D_2 f(a)]|}{|h|}$$
$$\leq \frac{|h^1||D_1 f(c_1) - D_1 f(a)| + |h^2||D_2 f(c_2) - D_2 f(a)|}{|h|.} \leq \frac{|h|(\dots)}{|h|}$$
$$= |D_1 f(c_1) - D_1 f(a)| + |D_2 f(c_2) - D_2 f(a)| \longrightarrow 0$$

Since ξ^2 between a^2 and a^2+h^2
ξ^1 " a^1 and a^1+h^1
as $(h^1, h^2) \to (0,0)$ $c_1 \to a$, $c_2 \to a$.
We are given that $D_1 f$, $D_2 f$ are continuous at a
$D_1 f(c_1) \to D_1 f(a)$, $D_2 f(c_2) \to D_2 f(a)$.

Definition:

If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ has partial derivatives $D_j f^i(x)$ $\forall x \in U$, U open, $a \in U$ and $D_j f^i$ is continuous at a we say f is continuously differentiable at a.

Example

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\quad x : \mathbb{R} \longrightarrow \mathbb{R}$

$\quad\quad\quad\quad\quad\quad y : \mathbb{R} \longrightarrow \mathbb{R}$

Define $g : \mathbb{R} \longrightarrow \mathbb{R}$ $\quad g(t) = f(x(t), y(t))$

$$\frac{dg}{dt}(t_0) = (g'(t_0)) = f'(x(t_0), y(t_0)) \cdot \begin{pmatrix} x'(t_0) \\ y'(t_0) \end{pmatrix}$$

$$= \left(\frac{\partial f}{\partial x}(x(t_0), y(t_0)), \frac{\partial f}{\partial y}(x(t_0), y(t_0)) \right) \cdot \begin{pmatrix} x'(t_0) \\ y'(t_0) \end{pmatrix}$$

$$\frac{dg}{dt}(t_0) = \frac{\partial f}{\partial x}(x_0, t_0) \cdot \frac{dx}{dt}(t_0) + \frac{\partial f}{\partial y}(x(t_0), y(t_0)) \cdot \frac{dy}{dt}(t_0)$$

$$\frac{dg}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\begin{tikzcd} & f \arrow[dl, dash, "\partial f/\partial x"'] \arrow[dr, dash, "\partial f/\partial y"] & \\ x \arrow[dr, dash, "dx/dt"'] & & y \arrow[dl, dash, "dy/dt"] \\ & t & \end{tikzcd}$$

Inverse Function Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with continuous derivative f', and assume $f'(a) \neq 0$.

By invareance principle $\exists$ interval J st $a \in J$, $\forall x \in J, f'(x) \neq 0$

Case 1 $f'(a) > 0$

On J, f is strictly increasing

$x, y \in J \quad x > y \implies f(x) > f(y)$

By MVT $\dfrac{f(x) - f(y)}{x - y} = f'(\xi) > 0$.

J is an interval

$I = f(J)$ is an interval by IVT

f is bijective from J to I

$f^{-1} : I \to J$, f is differentiable and

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

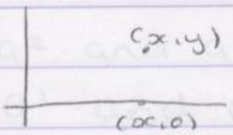
$$\lim_{h \to 0} \frac{f^{-1}(y+h) - f^{-1}(y)}{h} = \lim_{\delta \to 0} \frac{x + \delta - x}{f(x+\delta) - f(x)}$$

$$= \lim_{\delta \to 0} \frac{\delta}{f(x+\delta) - f(x)}$$

$$= \frac{1}{f'(x)}$$

What about $f : \mathbb{R}^2 \to \mathbb{R}^2$?

$f(x, y) = (x, 0)$



$$f'(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq 0$$

$\det f'(a) \neq 0 \implies f'(a)$ is invertible $\iff Df(a)$ is invertible linear map.

$f : \mathbb{R}^n \to \mathbb{R}^n$ $\quad a \in \mathbb{R}^n$ $\quad Df(a) : \mathbb{R}^n \to \mathbb{R}^n$ linear, $b = f(a)$

Assume we have found f^{-1}, $f \circ f^{-1} = \text{Id}$

Chain rule $\quad f'(f^{-1}(b)) \cdot (f^{-1})'(b) = I$

$$(f^{-1})'(b) = \cancel{\frac{1}{f'(f^{-1}(b))}}$$

$$= [f'(f^{-1}(b))]^{-1}$$

$Df(f^{-1}(b)) \circ (Df^{-1})(b) = \text{Id}$

$(Df^{-1})(b)$ is the inverse linear map to $Df(f^{-1}(b))$.

Theorem:

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable on an open set containing a and assume $\det f'(a) \neq 0$. Then

$\exists$ V open set, $a \in V$

$\exists$ W open set, $f(a) \in W$ st

$f : V \to W$ is bijective & st $f^{-1} : W \to V$ continuously diff.

and $\forall y \in W$ $\quad (f^{-1})'(y) = [f'(f^{-1}(y))]^{-1}$

(these are $n \times n$ matrices).

Example

$f : \mathbb{R}^2 \to \mathbb{R}^2$

$(z, w) = f(x, y) = (xy, x^2 + y^2)$ $\qquad z = xy \qquad w = x^2 + y^2$

$$f'(x, y) = \begin{pmatrix} y & x \\ 2x & 2y \end{pmatrix}$$

$\det f'(x, y) = 2y^2 - 2x^2$

$= 2(y+x)(y-x)$

$\det f'(x, y) \neq 0$ iff $y \neq \pm x$

$y = z/x \qquad w = x^2 + (z/x)^2$

$$w = x^2 + \frac{z^2}{x^2}$$

$$x^2 w = x^4 + z^2$$

$$x^4 - w x^2 + z^2 = 0$$

$$x^2 = t \qquad t^2 - wt + z^2 = 0$$

$$t = \frac{w \pm \sqrt{w^2 - 4z^2}}{2}$$

$$x = \pm \sqrt{\frac{w \pm \sqrt{w^2 - 4z^2}}{2}}$$

$$y = \frac{z}{\pm \sqrt{\frac{w \pm \sqrt{w^2-4z^2}}{2}}}$$

You should be able to differentiate if $w^2 - 4z^2 \neq 0$

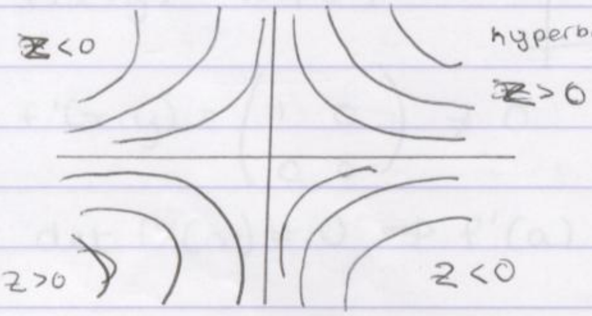
$$w^2 - 4z^2 = (x^2+y^2) - 4x^2y^2$$
$$= (x^2 - y^2)^2$$
$$= (x+y)^2 (x-y)^2$$

$$\begin{pmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial x}{\partial w} \end{pmatrix} = (f^{-1})'(z, w) \underset{\text{By Inverse Functn Thm}}{=} \begin{pmatrix} y & x \\ 2x & 2y \end{pmatrix}^{-1}$$

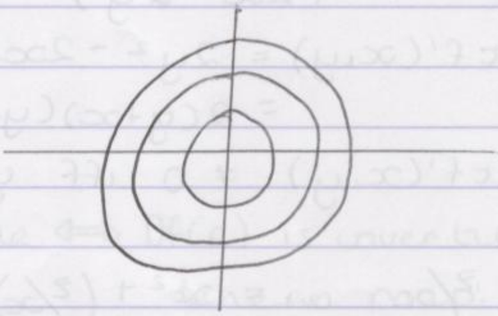
$$= \frac{1}{2(y^2 - x^2)} \begin{pmatrix} 2y & -x \\ -2x & y \end{pmatrix}$$

$$\frac{\partial x}{\partial z} = \frac{2y}{2(y^2 - x^2)} \qquad \frac{\partial x}{\partial w} = \frac{-x}{2(y^2 - x^2)} \qquad \text{etc.}$$

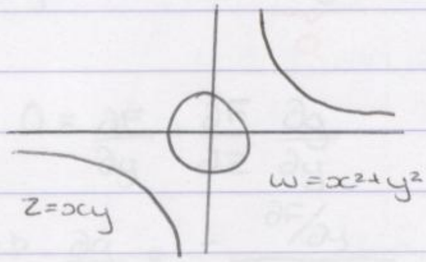
level curves of $z = xy$



$w = x^2 + y^2$

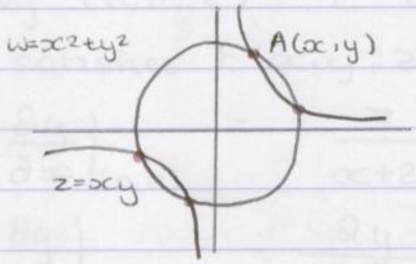


Consider (z, w) st.



(z, w) has no preimage.

Now consider

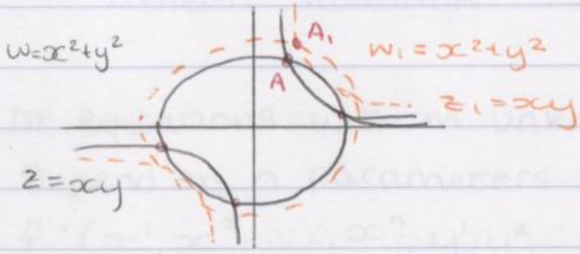


$A(x, y)$ $\quad x^2+y^2=w \quad z=xy$

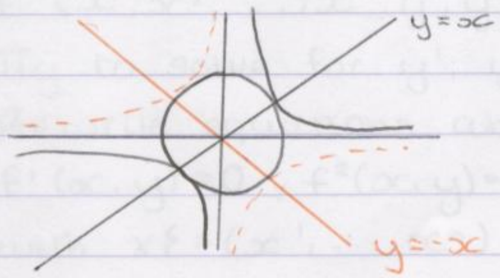
$(z, w)=f(x, y)$

$(x, y)=f^{-1}(z, w)$

Take z_1, w_1 close to z, w



Now consider



$w=x^2+y^2 \quad z=xy$

if $x=y$ the circle and the hyperbola meet tangentialy.

If we look at z_1, w_1 close to z, w it pushes into one of other cases

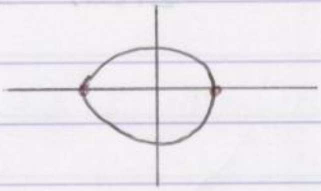
Implicit Function Theorem

Example

$x^2+y^2=1$

$y=g(x)$

$2x+2y\dfrac{dy}{dx}=0$



$\dfrac{dy}{dx}=\dfrac{dg}{dx}=-\dfrac{x}{y}\ ,\ y\neq 0$

Example

$y^2+xz+z^2-e^z-4=0$

Impossible to solve for z

$z=g(x,y)$

What are $\dfrac{\partial g}{\partial x}, \dfrac{\partial g}{\partial y}$?

Set $F(x,y,z)=y^2+xz+z^2-e^z-4$

$F(x,y,g(x,y))=0$

Differentiate in x

$\dfrac{\partial}{\partial x}F(x,y,g(x,y))=0$

$=\dfrac{\partial F}{\partial x}\underbrace{\dfrac{\partial x}{\partial x}}_{1}+\dfrac{\partial F}{\partial y}\underbrace{\dfrac{\partial y}{\partial x}}_{0}+\dfrac{\partial F}{\partial z}\dfrac{\partial z}{\partial x}=0$

$=\dfrac{\partial F}{\partial x}+\dfrac{\partial F}{\partial z}\dfrac{\partial g}{\partial x}=0$

$\dfrac{\partial g}{\partial x}=-\dfrac{\partial F/\partial x}{\partial F/\partial z}=-\dfrac{z}{x+2z-e^z}$

$$\frac{\partial F}{\partial y} = 0 = \frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial y}}_{0} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial y}}_{1} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y}$$

$$0 = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial g}{\partial y}$$

$$\Rightarrow \frac{\partial g}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z} = -\frac{2y}{x+2z-e^z}$$

eg $(0,e,2)$ $\quad e^2+0\cdot 2+2^2-e^2-4=0$

satisfies $F(x,y,z)=0$

$$\left.\frac{\partial g}{\partial x}\right|_{(0,e)} = -\frac{z}{x+2z-e^z} = -\frac{2}{0+2\cdot 2-e^2}$$

$$\left.\frac{\partial g}{\partial y}\right|_{(0,e)} = -\frac{2y}{x^2+2z-e^z} = \frac{-2e}{0+2\cdot 2-e^2}$$

General Situation

m equations with m unknowns : $y^1, y^2, \ldots y^m$

Depend on n parameters $x^1, x^2, \ldots, x^n$

$f^1(x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^m) = 0$

$f^2(x^1, x^2, \ldots, x^n, y^1, y^2, \ldots y^m) = 0$

$\vdots$

$f^m(x^1, x^2, \ldots, x^n, y^1, y^2, \ldots y^m) = 0$

Try to solve for $y^1, y^2, \ldots, y^m$

Rewrite equations above as

$f^1(x,y)=0$, $f^2(x,y)=0$, $\ldots$, $f^m(x,y)=0$

with $x = (x^1, \ldots, x^n)$ $\quad y = (y^1, \ldots y^m)$

Define $f(x,y) = (f^1(x,y), f^2(x,y), \ldots, f^m(x,y)) = 0 = \underbrace{(0,0,\ldots,0)}_{m\ 0s.}$

1

Let $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ st $f(a,b)=0$
When can we find for each $(x^1,\dots,x^n)$ near $\overset{a=}{}(a^1,\dots,a^n)$ a unique $y=(y^1,\dots,y^m)$ near $(b^1,\dots,b^m)=b$ such that
$f(x,y)=0$, $f(x^1,\dots,x^n,y^1,\dots y^m)=0$.

Theorem : Implicit function Theorem

$f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ continuously differentiable on an open set V, containing (a,b) $a \in \mathbb{R}^n$ $b \in \mathbb{R}^m$
Moreover $f(a,b)=0$
Consider the matrix $M = \left(D_{j+n} f^i(a,b)\right)_{j=1,\dots m}^{i=1,\dots m}$
Assume $\det M \neq 0$.
Then there exist two open sets $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$, $a \in A$, $b \in B$ st $\forall x \in A$ $\exists$ a unique $g(x) \in B$ st $f(x, g(x))=0$
Moreover $g: A \to B$ differentiable

Proof : increase the dimension of the target!

Define $F: \underset{\mathbb{R}^n \times \mathbb{R}^m}{\underset{\subseteq}{U}} \longrightarrow \mathbb{R}^n \times \mathbb{R}^m$

$F(x^1,x^2,\dots,x^n,y^1,y^2,\dots y^m) = (x^1,x^2,\dots,x^n,f^1(x,y),f^2(x,y),\dots,f^m(x,y))$
$F(x,y) = (x, f(x,y))$
F is continuously differentiable because $x^1,x^2,\dots,x^n$ are continuously differentiable and $f^1(x,y),\dots,f^m(x,y)$ are continuously differentiable (because $f(x,y)$ is continuously differentiable)

$F(a,b) = (a, f(a,b))$
$= (a, 0)$

Jacobian

$$\left(\begin{array}{cccc|ccc}
1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & & 0 & \cdots & 0 \\
\vdots & 0 & \ddots & \vdots & \vdots & & \vdots \\
\vdots & & & 0 & \vdots & & \vdots \\
0 & 0 & \cdots & 0\ 1 & 0 & \cdots & 0 \\
\hline
\frac{\partial f^1}{\partial x^1} & \frac{\partial f^1}{\partial x^2} & \cdots & \frac{\partial f^1}{\partial x^n} & \frac{\partial f^1}{\partial y^1} & \cdots & \frac{\partial f^1}{\partial y^m} \\
\frac{\partial f^2}{\partial x^1} & \frac{\partial f^2}{\partial x^2} & \cdots & \frac{\partial f^2}{\partial x^n} & \frac{\partial f^2}{\partial y^1} & \cdots & \frac{\partial f^2}{\partial f^m} \\
\vdots & & & & \vdots & & \\
\frac{\partial f^m}{\partial x^1} & \frac{\partial f^m}{\partial x^2} & \cdots & \frac{\partial f^m}{\partial x^n} & \frac{\partial f^m}{\partial y^1} & \cdots & \frac{\partial f^m}{\partial y^m}
\end{array}\right) = F'(a,b)$$

$$F'(a,b) = \left(\begin{array}{c|c} I_{n\times m} & 0_{n\times m} \\ \hline \begin{array}{c} x \\ \text{some matrix} \\ \text{of size } m\times n \end{array} & M_{m\times m} \end{array}\right)$$

$\det F'(a,b) = \det M \neq 0$

By the inverse function theorem $\exists$ open set W containing $F(a,b) = (a,0)$ and an open set containing (a,b), which I can take to be a rectangle $A\times B$, $a\in A$, $b\in B$, A open in $\mathbb{R}^n$, B open in $\mathbb{R}^m$

$F: A\times B \to W$ is bijective

$\exists\ h = F^{-1}: W \to A\times B$ st $F\circ h = \mathrm{Id}$

h is continuously differentiable

h must have the form $h(x,y) = (x, k(x,y))$ for some function $k(x,y)$, $k: W \to \mathbb{R}^m$

$W \to B$

k is continuously differentiable

$F(h(x,y)) = (x,y)$

$(x, f(x,k(x,y))) = (x,y)$

$f(x, k(x,y)) = y$

set $y=0$ $\quad f(x, k(x,0)) = 0$

The solution is $g(x) = \cancel{h}\, k(x,0)$

Example.

$f(x,y) = (xy, x^2+y^2) = (z,w)$

$f: \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} \partial x/\partial z & \partial x/\partial w \\ \partial y/\partial z & \partial y/\partial w \end{pmatrix} = \begin{pmatrix} \partial z/\partial x & \partial z/\partial y \\ \partial w/\partial x & \partial w/\partial y \end{pmatrix}^{-1}$$

$z = xy$ $\quad \cancel{\text{or}}, \; y = \dfrac{z}{x}, \quad x = g(z,w)$

$$w = x^2 + y^2 = x^2 + \frac{z^2}{x^2}$$

$wx^2 = x^4 + z^2$

$x^4 - wx^2 + z^2 = 0$

$4x^3 \dfrac{\partial x}{\partial z} - w2x\dfrac{\partial x}{\partial z} + 2z = 0$ (implicit diff. with respect to z)

$\dfrac{\partial x}{\partial z}(4x^3 - 2xw) = -2z$

$$\frac{\partial x}{\partial z} = \frac{-2z}{4x^3 - 2xw} = \frac{-xy}{x(2x^2-w)} = \frac{-y}{2x^2-w}$$

Valid for $\quad 2x^2 - w \neq 0$

$\quad 2x^2 - (x^2+y^2) = 0$

$\quad x^2 - y^2 \neq 0$

$\quad f'(x,y) \neq 0.$

$f(x,y)=0 \quad f(a,b)=0$ } set up of Implicit function theorem
$f(x,g(x))=0$ solving implicitly for y
$x \in \mathbb{R}^n,\ y \in \mathbb{R}^m \quad g: \mathbb{R}^n \to \mathbb{R}^m$.
$f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$

$i=1,\dots m \quad f^i(x^1,x^2,\dots,x^n,g^1(x^1,\dots x^n),g^2(x^1,\dots,x^n),\dots g^m(x^1,\dots x^n))=0$

How do we compute $D_j g^i$?

$D_j f^i(\dots)=0$

$$D_1 f^i \cancel{\frac{\partial x^1}{\partial x^j}}^{0} + D_2 f^i \cancel{\frac{\partial x^2}{\partial x^j}}^{0} + \dots + D_j f^i \overbrace{\frac{\partial x^j}{\partial x^j}}^{=1} + \dots + D_n f^i \cancel{\frac{\partial x^n}{\partial x^j}}^{0} + D_{n+1} f^i \frac{\partial g^1}{\partial x^j}$$
$$+ D_{n+2} f^i \frac{\partial g^2}{\partial x^j} + \dots + D_{n+m} f^i \frac{\partial g^m}{\partial x^j} = 0$$

$$D_{n+1} f^i \underline{\frac{\partial g^1}{\partial x^j}} + D_{n+2} f^i \underline{\frac{\partial g^2}{\partial x^j}} + \dots + D_{n+m} f^i \underline{\frac{\partial g^m}{\partial x^j}} = -D_j f^i$$

— unknowns.

$i=1,\dots m$.

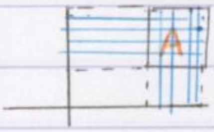
m equations.

Check det of coeffients is $\neq 0$.

$$\begin{bmatrix} D_{n+1} f^1 & D_{n+2} f^1 & \dots & D_{n+m} f^1 \\ D_{n+1} f^2 & D_{n+2} f^2 & \dots & D_{n+m} f^2 \\ \vdots & & & \\ D_{n+1} f^m & D_{n+2} f^m & \dots & D_{n+m} f^m \end{bmatrix} = M.$$ (from implicit function theorem)

Intergration

$f: A \rightarrow \mathbb{R}$, A is a rectangle in $\mathbb{R}^n$, $A = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$



Recall a partician P at $[a, b]$ is a collection of points $t_0, t_1, \dots, t_k$ with $a = t_0 < t_1 < t_2 \dots < t_k = b$.

A partition of the rectangle $[a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$ is a collection $\mathcal{P} = (P_1, P_2, \dots, P_n)$ with P_i a partition $[a_i, b_i]$, $i = 1, \dots, n$

Let f be bounded on the rectangle $[a_1, b_1] \times \dots \times [a_n, b_n]$
Let S be a subrectangle of the partition $\mathcal{P}$.

Definition :

$m_S(f) = \inf_{x \in S} f(x)$, $M_S(f) = \sup_{x \in S} f(x)$

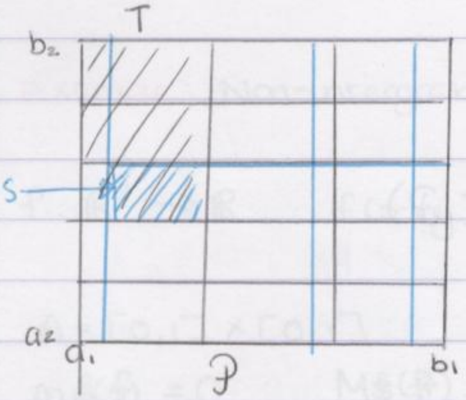
Lower Riemann Sum
$\mathcal{L}(f, \mathcal{P}) = \sum_S m_S(f) \cdot v(S)$ where $v(S)$ is the volume of the subrectangles
$S = [s_{i-1}, s_i] \times [t_{j-1}, t_j] \times \dots \times [r_{k-1}, r_k]$
$v(S) = (s_i - s_{i-1}) \cdot (t_j - t_{j-1}) \cdot \dots \cdot (r_k - r_{k-1})$.

Upper Riemann Sum
$U(f, \mathcal{P}) = \sum_S M_S(f) v(S)$

$\mathcal{L}(f, \mathcal{P}) \leq U(f, \mathcal{P})$

Refinement.

A refinement $\mathcal{P}'$ of the partition $\mathcal{P}$ is as follows (given S a subrectangle of $\mathcal{P}$). I can find a subrectangle T of $\mathcal{P}$ st $S \subset T$ and $T = \bigcup_{S \subset T} S$
S ~~from~~ $\mathcal{P}'$.



Lemma:

If $\mathcal{P}'$ is a refinement of $\mathcal{P}$

1. $\mathcal{L}(f,\mathcal{P}) \leq \mathcal{L}(f,\mathcal{P}')$
2. $U(f,\mathcal{P}) \geq U(f,\mathcal{P}')$

Proof 1. Let S be subrectangle of $\mathcal{P}'$ and T subrectangle of $\mathcal{P}$ st. $S \subset T$.

$m_S(f) \geq m_T(f)$

$m_S(f)v(S) \geq m_T(f)v(S)$ sum over all $S \subset T$, S for $\mathcal{P}'$

$$\sum_{S\subset T} m_S(f)v(S) \geq \sum_{S\subset T} m_T(f)v(S)$$

$$= m_T(f)v(T)$$

$$\sum_{T}\sum_{S\subset T} m_S(f)v(S) \geq \sum_{T} m_T(f)v(T)$$

$$= \mathcal{L}(f,\mathcal{P})$$

$$\underbrace{\sum_{S \text{ of } \mathcal{P}'} m_S(f)v(S)}_{= \mathcal{L}(f,\mathcal{P}')} \geq \mathcal{L}(f,\mathcal{P})$$

Lemma:

For any two partitions $\mathcal{P},\mathcal{P}'$, $\mathcal{L}(f,\mathcal{P}) < U(f,\mathcal{P}')$

Proof: Take $\mathcal{P}''$ a refinement of both $\mathcal{P}$ and $\mathcal{P}''$

$\mathcal{L}(f,\mathcal{P}) \leq \mathcal{L}(f,\mathcal{P}'') \leq U(f,\mathcal{P}'') \leq U(f,\mathcal{P}')$.

Definition:

The lower Riemann intergral $\underline{\int_A} f = \sup_{\mathcal{P}} \mathcal{L}(f,\mathcal{P})$

The upper Riemann intergral $\overline{\int_A} f = \inf_{\mathcal{P}} U(f,\mathcal{P})$

f is called intergralable if $\underline{\int_A} f = \overline{\int_A} f$ and $\int_A f = \underline{\int_A} f = \overline{\int_A} f$.

Theorem: Riemann intergrability criterion.

f is intergrable over the rectangle A iff $\forall \varepsilon > 0 \ \exists$ partition $\mathcal{P}$ of A st $U(f,\mathcal{P}) - \mathcal{L}(f,\mathcal{P}) < \varepsilon$.

Proof: $\Leftarrow$ $\inf(U(f,\mathcal{P}) - \mathcal{L}(f,\mathcal{P})) = 0$

$\inf U(f,\mathcal{P}) - \sup \mathcal{L}(f,\mathcal{P}) = 0$

$\overline{\int_A} f - \underline{\int_A} f = 0.$

$\Rightarrow$ Assume $\underline{\int_A} f = \overline{\int_A} f$

Fix $\varepsilon > 0$

Since $\underline{\int_A} f = \sup_{\mathcal{P}} \mathcal{L}(f,\mathcal{P})$ $\exists \mathcal{P}$ st $\underline{\int_A} f - \varepsilon/2 < \mathcal{L}(f,\mathcal{P})$

Since $\overline{\int_A} f = \inf_{\mathcal{P}} U(f,\mathcal{P})$, $\exists \mathcal{P}'$ st $\overline{\int_A} f + \varepsilon/2 > U(f,\mathcal{P}')$

Take $\mathcal{P}''$ a common refinement of $\mathcal{P}$ and $\mathcal{P}'$

$\overline{\int_A} f + \varepsilon/2 > U(f,\mathcal{P}'') \geq \mathcal{L}(f,\mathcal{P}'') > \underline{\int_A} f - \varepsilon/2$

$U(f,\mathcal{P}'') - \mathcal{L}(f,\mathcal{P}'') < \left(\overline{\int_A} f + \varepsilon/2\right) - \left(\underline{\int_A} f - \varepsilon/2\right) = \varepsilon.$

Example: Non-intergrable ~~[illegible]~~ function.

$f : \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q}. \end{cases}$

$A = [0,1] \times [0,1]$

$m_S(f) = 0 \qquad M_S(f) = 1$

$L(f, P) = 0 \qquad U(f, P) = 1$

If $C \subset \mathbb{R}^n$ define the characteristic function of C to be

$X_C(x) = \begin{cases} 1 & x \in C \\ 0 & x \notin C \end{cases}$

If f is bounded on $\overline{C}$ and C is contained in a rectangle A we define $\int_C f = \int_A f X_C$.

How to compute intergrals?

Use Fubini's theorem.

$f : [a,b] \times [c,d] \to \mathbb{R}$

fix x. consider $g_x : [c,d] \to \mathbb{R}$

$g_x(y) = f(x,y)$

$I(x) = \int_c^d g_x \, dy = \int_c^d f(x,y) \, dy$

$\int_a^b I(x) \, dx = \int_a^b \left(\int_c^d f(x,y) \, dy \right) dx.$

or fix y, define $h_y(x) = f(x,y)$

$h_y : [a,b] \to \mathbb{R}$

$\int_a^b h_y(x) \, dx = \int_a^b f(x,y) \, dx = J(y)$

$\int_c^d J(y) \, dy = \int_c^d \left(\int_a^b f(x,y) \, dx \right) dy.$

Theorem : Fubini's Theorem.

Let A be a rectangle in $\mathbb{R}^n$
Let B be a rectangle in $\mathbb{R}^m$
Let $f : A \times B \to \mathbb{R}$ be intergrable over the rectangle $A \times B$
Let $\mathcal{L}(x) = \underline{\int_B} f(x,y)\,dy$
$U(x) = \overline{\int_B} f(x,y)\,dy$
Defined for all $x \in A$, because $\underline{\int}, \overline{\int}$ always exist.

Then $\int_A \mathcal{L}(x) = \int_A U(x) = \int_{A\times B} f = \int_A \left(\overline{\int_B} f(x,y)\,dy \right) dx$ exists.

Remarks.

1. If $\forall x \in A$ $\int_B f(x,y)\,dy$ exists ie $\mathcal{L}(x) = U(x)$ the Fubini reads as

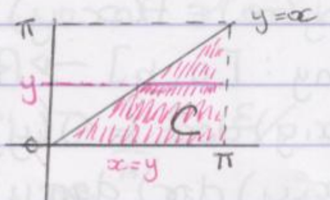
$$\int_{A\times B} f = \int_A \left(\int_B f \right) = \int_A \left(\int_B f(x,y)\,dy \right) dx$$

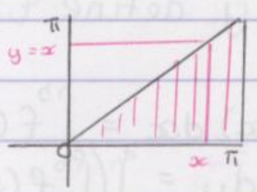
2. Similarly define $\mathcal{L}(y) = \underline{\int_A} f(x,y)\,dx$, $U(y) = \overline{\int_A} f(x,y)\,dx$

Fubini says $\mathcal{L}(y), U(y)$ are intergrable over B and
$$\int_{A\times B} f = \int_B \mathcal{L}(y)\,dy = \int_B \left(\underline{\int_A} f(x,y)\,dx \right) dy = \int_B U(y)\,dy = \int_B \left(\overline{\int_A} f(x,y)\,dx \right) dy$$

Example

$$\int_0^{\pi} \left(\int_y^{\pi} \frac{\sin x}{x}\,dx \right) dy$$





$$\overset{\text{Fubini}}{=} \int_{[0,\pi]\times[0,\pi]} \frac{\sin x}{x} \chi_C(x,y) \overset{\text{Fubini}}{=} \int_0^{\pi} \left(\int_0^{x} \frac{\sin x}{x}\,dy \right) dx$$

$$= \int_0^{\pi} \frac{\sin x}{x}(x-0)\,dx = \int_0^{\pi} \sin x\,dx$$

$= -\cos x \Big|_0^{\pi}$
$= -\cos\pi + \cos 0$
$= 1 + 1 = 2.$

Fubini's Theorem

Let A be a rectangle in $\mathbb{R}^n$, B a rectangle in $\mathbb{R}^m$.
$f: A\times B \longrightarrow \mathbb{R}$ be intergrable.
Define $g_x : B \longrightarrow \mathbb{R}$ by $g_x(y) = f(x,y)$ $\forall y \in B$, $\forall x \in A$ and let

$$\left.\begin{aligned} \mathcal{L}(x) &= \underline{\int_B} g_x = \underline{\int_B} f(x,y)\,dy \\ \mathcal{U}(x) &= \overline{\int_B} g_x = \overline{\int_B} f(x,y)\,dy \end{aligned}\right\} \text{ exist } \forall x \in A$$

Then $\mathcal{L}(x)$, $\mathcal{U}(x)$ are intergrable over A and

$$\int_A \mathcal{L}(x)\,dx = \int_A \left(\underline{\int_B} f(x,y)\,dy \right) dx = \int_A \mathcal{U}(x)\,dx = \int_A \left(\overline{\int_B} f(x,y)\,dy \right) dx = \int_{A\times B} f.$$

Proof: Let P_A be a partition of A, P_B a partition of B.
S_A a subrectangle of P_A, S_B a subrectangle of P_B
Then the rectangle $S_A \times S_B$ given a partition P of $A\times B$.
We will prove

$$L(f,P) \underset{1}{\leq} L(\mathcal{L}, P_A) \underset{2}{\leq} U(\mathcal{L}, P_A) \underset{3}{\leq} U(\mathcal{U}, P_A) \underset{4}{\leq} U(f,P)$$

Since f is intergrable over $A\times B$, given $\varepsilon > 0$ Riemann's intergrability criterion given a partition P of $A\times B$ st $U(f,P) - L(f,P) < \varepsilon$.
Then P defines P_A, P_B partitions of A and B respectively
By the inequality above

$$U(\mathcal{L}, P_A) - L(\mathcal{L}, P_A) < \varepsilon.$$

By Riemann intergrability criterion $\mathcal{L}$ is intergrable over A since $\sup_P L(f,P) = \inf_P U(f,P) = \int_{A\times B} f$

$$\Rightarrow \int_A \mathcal{L}(x)\,dx = \sup_{P_A} L(\mathcal{L}, P_A) = \inf_{P_A} U(\mathcal{L}, P_A) = \int_{A\times B} f$$

$L(f,P) \leq L(\mathcal{L}, P_A)$
$\sup_P L(f,P) \leq \sup_{P_A} L(\mathcal{L}, P_A)$
Similarly for U.

Similarly ~~for~~ with $\mathcal{U}(x)$

Proof of inequality.

2. $L(\alpha, P_A) \leq U(\alpha, P_A)$

always true for a function α, partition P_A that lower riemann sum $\leq$ upper riemann sum

3. $U(\alpha, P_A) \leq U(\mathcal{U}, P_A)$

$\alpha(x) = \underline{\int_B} f(x,y)\,dy$, $\mathcal{U} = \overline{\int_B} f(x,y)\,dy$

$\Rightarrow \alpha(x) \leq \mathcal{U}(x) \Rightarrow U(\alpha, P_A) \leq U(\mathcal{U}, P_A)$ larger function has larger riemann sum.

4. Proved similarly to 1.

1. $L(f,P) \leq L(\alpha, P_A)$

take $S_A \times S_B$ of P

Let $x \in S_A$ $\quad \underset{S_A \times S_B}{m}(f) \leq \underset{\{x\} \times S_B}{m}(f)$ (inf over a smaller set is larger)

Multiply with $v(S_B)$ and sum over S_B's of P_B

$$\sum_{S_B} \underset{S_A\times S_B}{m}(f)\, v(S_B) \leq \sum_{S_B} \underset{\{x\}\times S_B}{m}(f)\, v(S_B) = L(g_x, P_B)$$

$$\leq \underline{\int_B} g_x = \alpha(x)$$

Take inf over $x \in S_A$

$$\sum_{S_B} \underset{S_A\times S_B}{m}(f)\, v(S_B) \leq \inf_{x \in S_A} \alpha(x) = \underset{S_A}{m}(\alpha)$$

$$\sum_{S_B} \underset{S_A\times S_B}{m}(f)\, v(S_B) \leq \underset{S_A}{m}(\alpha)$$

multiply with $v(S_A)$ sum over S_A

$$\underbrace{\sum_{S_A}\sum_{S_B} \underset{S_A\times S_B}{m}(f)\, \underbrace{v(S_B)v(S_A)}_{v(S_A\times S_B)}}_{L(f,P)} \leq \underbrace{\sum_{S_A} \underset{S_A}{m}(\alpha)\, v(S_A)}_{L(\alpha, P_A)}$$

Warning!

Let $f(x,y) = \begin{cases} 1 & \text{if } x \neq \pi \\ 0 & \text{if } x = \pi, y \notin \mathbb{Q} \\ 1 & \text{if } x = \pi, y \in \mathbb{Q} \end{cases}$

$$\int_{A\times B} f = \int_{A\times B} 1 = (2\pi)^2$$

Notice $g_\pi(y) = \begin{cases} 1 & y \in \mathbb{Q} \\ 0 & y \notin \mathbb{Q} \end{cases}$ is not intergrable

Change of Variables

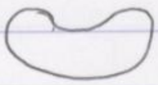
Let $A \subseteq \mathbb{R}^n$ be open, $g: A \to \mathbb{R}^n$ be injective continuously differentiable with $\det g'(x) \neq 0 \ \forall x \in A$.
Let $f: g(A) \to \mathbb{R}$ be intergrable. Then we have the change of variable formula:

$$\int_{g(A)} f = \int_A (f \circ g) |\det g'(x)| \, dx$$

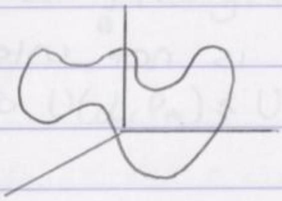
Manifolds

M K-dim manifold in $\mathbb{R}^n$

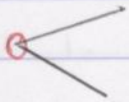
eg. • 1-dim $\mathbb{R}^2$



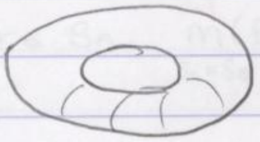
• 1-dim in $\mathbb{R}^3$



Not manifold



• 2-dim surface in $\mathbb{R}^3$



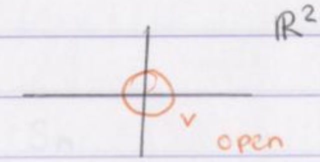
doughnut /torus

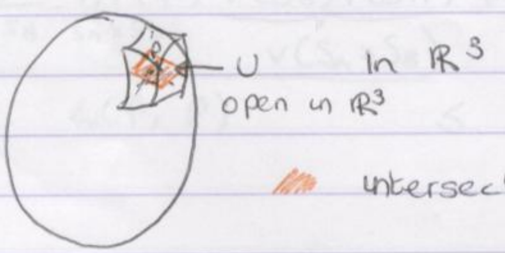
$S^2 \subseteq \mathbb{R}^3$

ellipsoids.

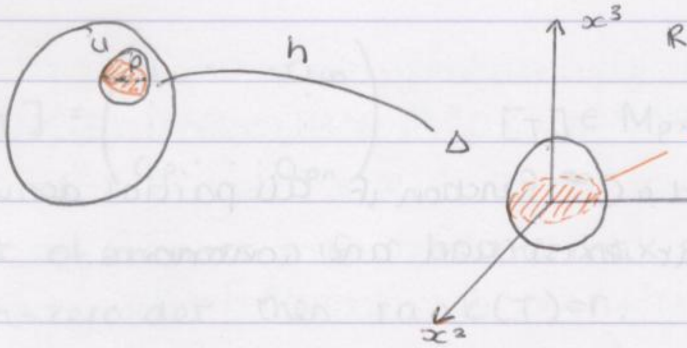
want

to look like





intersection.



Definition:

Let U, V be open sets in $\mathbb{R}^n$, $h: U \rightarrow V$ be a bijection, which is differentiable. (all partials of all orders exist and are continuous) and such $h^{-1}: V \rightarrow U$ is also differentiable (all partials of all orders exist and are continuous)
Then we call h a differeomorphism from U to V

Definition:

A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is called a C^∞ function if all partial derivatives of all orders of all component exist and are continuous.

Definition:

Let U, V be open sets in $\mathbb{R}^n$. A C^∞ function $h: U \to V$ bijective and $h^{-1}: V \to U$ is also C^∞, is called a diffeomorphism from U to V.

Definition:

A set M is a K-dim manifold in $\mathbb{R}^n$ if the following condition (M) holds for every $x \in M$:

(M) there exist two open sets U, V of $\mathbb{R}^n$, $x \in U$ and a diffeomorphism $h: U \to V$ st $h(U \cap M) = \{y \in V \text{ st } y^{k+1} = y^{k+2} = \dots = y^n = 0\}$.

Theorem 1: Handout 3/4

Let $g: \mathbb{R}^n \to \mathbb{R}^p$, $n \geq p$ be a C^∞ function

Set $M = g^{-1}(0) = \{x \in \mathbb{R}^m, g(x) = 0 \in \mathbb{R}^p\}$.

If $\forall x \in M$ the rank of $g'(x)$ is p then $M = g^{-1}(0)$ is an $(n-p)$-dim manifold in $\mathbb{R}^n$

Remainder from Linear Algebra

$T: \mathbb{R}^n \to \mathbb{R}^p$ linear transformation

rank $(T) = \dim T(\mathbb{R}^n) \leq p$

$[T]$ = max number of LI of rows or columns.

rank $(T) \leq \min(n, p)$

$$[T] = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{p1} & \cdots & a_{pn} \end{pmatrix} \qquad [T] \in M_{p \times n}$$

det of minors, if r is the max size of an $r \times r$ minor with non-zero det then $\operatorname{rank}(T) = r$.

~~Theorem~~

Examples

1. $S^2 = \{(x, y, z) \in \mathbb{R}^3 \; x^2+y^2+z^2=1\}$ 2 dim manifold in $\mathbb{R}^3$

$g : \mathbb{R}^3 \to \mathbb{R}$, $g(x,y,z) = x^2+y^2+z^2-1$

$S^2 = g^{-1}(0)$

$g'(x,y,z) = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)$

1×3

$= (2x, 2y, 2z)$ rank can be 0 or 1

$\underset{n-p}{2} = \underset{n}{3} - \underset{p}{1}$

Aim to show $\operatorname{rank} g' = 1$ on $M = g^{-1}(0)$

$\operatorname{rank} g' = 0 \iff 2x = 2y = 2z = 0$

$\iff (x,y,z) = (0,0,0)$

but $(0,0,0) \in g^{-1}(0)$ because $g(0,0,0) = 0^2+0^2+0^2-1 = -1$

2. $S^n = \{(x^1, x^2, \ldots, x^n, x^{n+1}),\ (x^1)^2+(x^2)^2+\ldots+(x^{n+1})^2 = 1\}$

is an n-dim manifold in $\mathbb{R}^{n+1}$

$g : \mathbb{R}^{n+1} \to \mathbb{R}^1$ $\quad g(x^1, \ldots, x^{n+1}) = (x^1)^2+(x^2)^2+(x^{n+1})^2-1$

$S^n = g^{-1}(0)$

$g'(x^1, x^2, \ldots, x^{n+1}) = \left(\frac{\partial g}{\partial x^1}, \frac{\partial g}{\partial x^2}, \ldots, \frac{\partial g}{\partial x^{n+1}}\right)$

$= (2x^1, 2x^2, \ldots, 2x^{n+1})$

$\operatorname{rank} g' = 0 \iff 2x^1 = 2x^2 = \ldots = 2x^{n+1} = 0$

$\iff x^1 = x^2 = \ldots = x^{n+1} = 0$

but $(0, 0, \ldots 0) \notin S^n$

3. Hyperbolic space

$\mathbb{H}^n = \{ (x^1, \dots, x^n, x^{n+1}) \in \mathbb{R}^{n+1} \ x^1 > 0 \ (x^1)^2 - [(x^2)^2 + (x^3)^2 + \dots + (x^{n+1})^2] = 1 \}.$

$g(x^1, \dots, x^n, x^{n+1}) = (x^1)^2 - [(x^2)^2 + (x^3)^2 + \dots + (x^{n+1})^2] - 1$

$\mathbb{H}^n = g^{-1}(0) \qquad g : A \to \mathbb{R} \qquad A = \{x \in \mathbb{R}^{n+1}, x^1 > 0\}.$

$g'(x^1, \dots, x^n, x^{n+1}) = (2x^1, -2x^2, -2x^3, \dots, -2x^{n+1})$

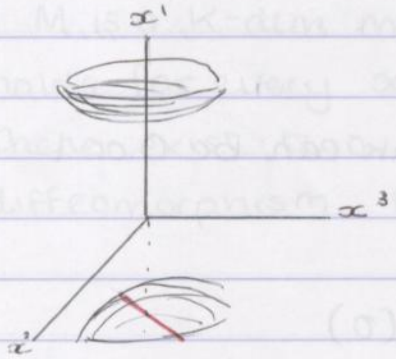
$\operatorname{rank} g' = 0 \iff x^1 = x^2 = \dots = x^{n+1} = 0$

but $(0, 0, \dots 0) \notin \mathbb{H}^n$

$\operatorname{rank} g' = 1$ on $g^{-1}(0)$ so by theorem $g^{-1}(0)$ is an $(n+1)-1$ manifold in $\mathbb{R}^{n+1}$

eg. $n = 2 \quad (x^1)^2 - (x^2)^2 - (x^3)^2 = 1 \quad x^1 > 0$

$\mathbb{R}^3$



Fix $x^2 = 0 \quad (x^1)^2 - (x^3)^2 = 1$

Fix $x^3 = 0 \quad (x^1)^2 - (x^2)^2 = 1$

4. Ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad a, b, c > 0$

$g(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$

$g'(x, y, z) = \left(\frac{2x}{a}, \frac{2y}{b}, \frac{2z}{c} \right)$

$g'(x, y, z) = 0 \iff x = y = z = 0$

but $(0, 0, 0)$ does not belong to ellipsoid.

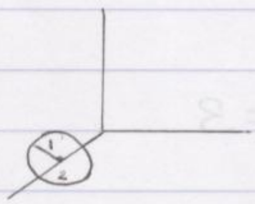
5. The graph of a differentiable function
$f: U \to \mathbb{R} \quad U \subseteq \mathbb{R}^2$
$M = \{(x, y, z) \in \mathbb{R}^3, z = f(x, y)\}$ (Monge Patch)
is a 2dim manifold in $\mathbb{R}^3$
$g(x, y, z) = f(x, y) - z$
$g'(x, y, z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right) \neq 0 \Rightarrow \text{rank } g' = 1$

Theorem 2

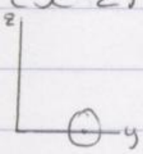
M is a K-dim manifold in $\mathbb{R}^n$ if and only if for every $x \in M$ the following condition holds
(C) There exists an open sets $W \subseteq \mathbb{R}^k$ $x \in U$ and an open set $U \subseteq \mathbb{R}^n$ and $f: W \to \mathbb{R}^n$ differentiable, injective st
i $f(W) = U \cap M$
ii $\text{rank } f'(y) = K \quad \forall y \in W$.
iii $f^{-1}: U \cap M \to W$ is continuous.

Examples

1. 2-dim torus 

Consider  (rotating around z-axis gives torus)

$(x-2)^2 + z^2 = 1$



$(y-2)^2 + z^2 = 1$

cylindrical coordinates
$z = \sin\phi$
$r - 2 = \cos\phi$

$f(\theta,\phi) = ((2+\cos\phi)\cos\theta, (2+\cos\phi)\sin\theta, \sin\phi)$

$\theta \in (-\pi, \pi)$

$\varphi \in (-\pi, \pi)$

For theorem 2 take $U = \mathbb{R}^3$

$f(W) = U \cap M$

$f(W) = M$

$f : \underset{\mathbb{R}^2}{\overset{\cap}{W}} \longrightarrow \mathbb{R}^3$

$$f'(\theta,\varphi) = \begin{pmatrix} (2+\cos\varphi)(-\sin\theta) & -\sin\varphi\cos\theta \\ (2+\cos\varphi)\cos\theta & -\sin\varphi\sin\theta \\ 0 & \cos\varphi \end{pmatrix}$$

Does it have rank 2 on $W = (-\pi,\pi)\times(-\pi,\pi)$

2×2 minor $\begin{vmatrix} (2+\cos\varphi)(-\sin\theta) & -\sin\varphi\cos\theta \\ (2+\cos\varphi)(\cos\theta) & -\sin\varphi\sin\theta \end{vmatrix}$

$= -(2+\cos\varphi)(\sin\varphi) \begin{vmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{vmatrix}$

$= (2+\cos\varphi)\sin\varphi \neq 0$ iff $\sin\varphi = 0 \Leftrightarrow \varphi = 0$

rank $f' = 2$ whenever $\varphi \neq 0$.

when $\varphi = 0$ $f'(\theta, 0) = \begin{pmatrix} -3\sin\theta & 0 \\ 3\cos\theta & 0 \\ 0 & 1 \end{pmatrix}$

If $\theta = 0$ use $\begin{vmatrix} 3\cos\theta & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3$

$\theta \neq 0$ use $\begin{vmatrix} -3\sin\theta & 0 \\ 0 & 1 \end{vmatrix} = -3\sin\theta \neq 0$ on $\theta \in (-\pi,\pi)$
$\theta \neq 0$.

General Example

Any nice surface of rotation is a 2-dim manifold in $\mathbb{R}^3$

$\gamma(t) = (r(t), z(t)$ $t \in (a,b)$

γ does not have self intersections

$r(t) > 0$

γ is differentiable $\gamma'(t) = (r'(t), z'(t)) \neq 0$ $\forall t \in (a,b)$

then we can rotate around the z-axis, we get the surface

$f(t,\theta) = (r(t)\cos\theta, r(t)\sin\theta, z(\theta))$ $t \in (a,b)$ $\theta \in (-\pi, \pi)$

$$f'(t,\theta) = \begin{bmatrix} r'\cos\theta & -r\sin\theta \\ r'\sin\theta & r\cos\theta \\ z' & 0 \end{bmatrix}$$

$$\begin{vmatrix} r'\cos\theta & -r\sin\theta \\ r'\sin\theta & r\cos\theta \end{vmatrix} = r \cdot r'$$ since $r > 0$, $r' \neq 0$

$\Rightarrow$ rank 2 if $r' = 0$, $z' \neq 0$.

Dual Spaces

V is a n dim vector space

A linear functional, f is a linear transformation, $f: V \to \mathbb{R}$

$f(\lambda x + y) = \lambda f(x) + f(y) \quad \forall x, y \in V, \; \forall \lambda \in \mathbb{R}$

Definition:

$V^* = \{f : V \to \mathbb{R}, f \text{ linear functionals}\} \ni f, g$

Define $(f+g)(x) = f(x) + g(x) \quad \forall x \in V, \quad f+g : V \to \mathbb{R}$

Given $\lambda \in \mathbb{R}$, $(\lambda f)(x) = \lambda f(x) \qquad \lambda f : V \to \mathbb{R}$.

Check $f+g \in V^*$

$\lambda \in \mathbb{R}, \; x, y \in V$

$$\begin{aligned}(f+g)(\lambda x + y) &= f(\lambda x + y) + g(\lambda x + y) \\ &= \lambda f(x) + f(y) + \lambda g(x) + g(y) \\ &= \lambda(f(x) + g(x)) + (f(y) + g(y)) \\ &= \lambda (f+g)(x) + (f+g)(y)\end{aligned}$$

Example

$V = P_{n-1}[x]$, fix $x_0 \in \mathbb{R}$

$f(p(x)) = p(x_0$

Show $f \in V^*$

Proposition:

$\dim V^* = \dim V$

Proof: We have $\{v_1, v_2, \dots v_n\}$ basis of V

$\forall i = 1, \dots n$ define $\varphi_i : V \to \mathbb{R}$ as follows

Given $x \in V$, $x = x^1 v_1 + x^2 v_2 + \ldots + x^n v_n$, $x^i \in \mathbb{R}$.

$\varphi_i(x) = x^i$

φ_i is a linear function

If $y = y^1 v_1 + y^2 v_2 + \ldots + y^n v_n$

$\lambda \in \mathbb{R}$. $\lambda x + y = (\lambda x^1 + y^1) v_1 + \ldots + (\lambda x^n + y^n) v_n$

$\varphi_i(\lambda x + y) = \lambda x^i + y^i$

$= \lambda \varphi_i(x) + \varphi_i(y)$

$\varphi_i(v_j) = \delta_{ij}$

$\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ is a basis for V^*?

They span and are L.I.

Given $f \in V^*$, define $a_i \in \mathbb{R}$, $f(v_i) = a_i \in \mathbb{R}$.

We will show $f = a^1 \varphi_1 + a^2 \varphi_2 + \ldots + a^n \varphi_n$

If f and $a^1 \varphi_1 + \ldots + a^n \varphi_n$ agree on the basis $\{v_1 \ldots v_n\}$ this is true

$f(v_k) = a_k$

$(a_1 \varphi_1 + \ldots + a^n \varphi_n)(v_k) = a^1 \varphi_1(v_k) + \ldots + a^k \varphi(v_k) + \ldots + a^n \varphi(v_n)$

$= a^k$

$b^1 \varphi_1 + b^2 \varphi_2 + \ldots + b^n \varphi = 0 \overset{?}{\Rightarrow} b^k = 0 \;\forall k.$

Apply to basis vector v_k

$(b^1 \varphi_1 + \ldots + b^n \varphi_n)(v_k) = b^1 \cdot 0 + b^2 \cdot 0 + \ldots + b^k \cdot 1 + b^{k+1} \cdot 0 = b^k$

Multilinear Algebra

$f: V \to \mathbb{R}$, linear map, linear functional

$V^* = \{f : V \to \mathbb{R}$ linear functional$\}$ dual space of V

If $\{v_1, v_2, \dots v_n\}$ is basis of V, then $\{\varphi_1, \varphi_2, \dots \varphi_n\}$ is a basis of V^*, $\varphi_i(v_j) = \delta_{ij}$

Exercise

$V = P_n[x]$ with basis $\{1, x, \dots x^n\}$ of V

$f : V \to \mathbb{R}$, $f(p) = p(3)$

Write f as a linear combination of dual basis.

Let V be vs over $\mathbb{R}$.

We define $V^k = \underbrace{V \times V \times V \times \dots \times V}_{k \text{ times}}$ to be

$V^k = \{(v_1, \dots v_k) \ v_i \in V\}$

This is a vector space with operations.

$(v_1, v_2, \dots v_k) + (w_1, w_2, \dots w_k) = (v_1 + w_1, v_2 + w_2, \dots, v_k + w_k)$

$\lambda(v_1, v_2 \dots v_k) = (\lambda v_1, \lambda v_2 \dots \lambda v_k)$

Check 8 properties so that V^k is vs $/\mathbb{R}$.

$T: V^k \to \mathbb{R}$ is called multilinear if for all $i \in \{1, 2, \dots k\}$

$T(v_1, v_2, \dots, v_{i-1}, v_i + v_i', v_{i+1}, \dots v_k) = T(v_1, v_2, \dots v_{i-1}, v_i, v_{i+1}, \dots v_k)$
$+ T(v_1, v_2 \dots v_{i-1}, v_i', v_{i+1}, \dots v_k)$

$\forall v_1, v_2 \dots v_k, v_i' \in V$.

$\lambda \in \mathbb{R}$

$T(v_1, v_2, \dots v_{i-1}, \lambda v_i, v_{i+1}, \dots v_k) = \lambda T(v_1, v_2, \dots v_{i-1}, v_i, v_{i+1}, \dots v_k)$.

A T like this is called a k-tensor on V

Define: $\mathcal{T}^k(V) = T : V^k \to \mathbb{R}$, k-multilinear.

Example

$T(v_1+v_2, w) = T(v_1, w) + T(v_2, w)$
$T(\lambda v_1, w) = \lambda T(v_1, w)$
$T(v_1, w_1+w_2) = T(v_1, w_1) + T(v_1, w_2)$
$T(v, \lambda w) = \lambda T(v, w)$
bilinear form, $k=2$.

Def

T is a symmetric k-tensor $\forall v_1, \dots v_i \in V$
$T(v_1, v_2, \dots, v_i, v_{i+1}, \dots, v_j \dots v_k)$
$= T(v_1, v_2 \dots v_j, \dots v_i, \dots v_k)$

Def.

T is an alternating k-tensor if
$T(v_1, v_2 \dots v_i, \dots v_j \dots v_k) = -T(v_1, v_2 \dots v_j, \dots v_i \dots v_k)$

Eg. $\mathbb{R}^2 = V$ $\quad \mathbb{R}^2 \times \mathbb{R}^2 = V^2$
$T(v_1, v_2) = v_1^1 v_2^2 - v_2^1 v_1^2$
$v_1 = (v_1^1, v_1^2)$, $v_2 = (v_2^1, v_2^2)$

$$\begin{vmatrix} \lambda v_1 + v_1' \\ v_2 \end{vmatrix} = \lambda \begin{vmatrix} v_1 \\ v_2 \end{vmatrix} + \begin{vmatrix} v_1' \\ v_2 \end{vmatrix}$$

$$\begin{vmatrix} v_1 \\ \lambda v_2 + v_2' \end{vmatrix} = \lambda \begin{vmatrix} v_1 \\ v_2 \end{vmatrix} + \begin{vmatrix} v_1 \\ v_2' \end{vmatrix}$$

$$\begin{vmatrix} v_1 \\ v_2 \end{vmatrix} = - \begin{vmatrix} v_2 \\ v_1 \end{vmatrix}$$

det on k matrix, (as functions of k vectors in $\mathbb{R}^k$ is an alternating ~~then~~ k-tensor).

If $T, S \in \mathcal{J}^k(V)$ we define
$(T+S)(v_1, v_2, \dots v_k) = T(v_1, \dots v_k) + S(v_1 \dots v_k)$
Ex show $(T+S) \in \mathcal{J}^k(V)$

Similarly $\lambda \in \mathbb{R}$, $\lambda T \in \mathcal{T}^k(V)$

$(\lambda T)(v_1, v_2 \ldots v_k) = \lambda T(v_1, \ldots v_k) \quad \forall v_1 \ldots v_k \in V$

Let $T \in \mathcal{T}^k(V)$, $S \in \mathcal{T}^l(V)$ $\quad k, l \in \mathbb{N}$

Define: $T \otimes S \in \mathcal{T}^{k+l}(V)$

$(T \otimes S)(v_1, v_2 \ldots v_k, v_{k+1}, \ldots v_{k+l}) \quad v_i \in V$

$= T(v_1 \ldots v_k) \cdot S(v_{k+1}, \ldots, v_{k+l})$

real number multiplication

$S \otimes T \in \mathcal{T}^{l+k}(V)$

$S \otimes T \neq T \otimes S$ in general.

Properties:

1. $T \otimes S \in \mathcal{T}^{k+l}(V)$ - homework
2. $(S_1 + S_2) \otimes T = S_1 \otimes T + S_2 \otimes T$
3. $S \otimes (T_1 + T_2) = S \otimes T_1 + S \otimes T_2$
4. $(\lambda S) \otimes T = \lambda (S \otimes T) = S \otimes (\lambda T)$
5. $(S \otimes T) \otimes U = S \otimes (T \otimes U)$
6. $\mathcal{T}^1(V) = V^*$

Theorem:

Let $i_1 \ldots i_k \in \{1, 2, \ldots n\}$

V has basis $\{v_1, v_2 \ldots v_n\}$ $\dim V = n$

Let $\{\varphi_1, \ldots \varphi_n\}$ be the dual basis of V^* $\varphi_i(v_j) = \delta_{ij}$

Consider $\varphi_{i_1} \otimes \varphi_{i_2} \otimes \ldots \otimes \varphi_{i_k}$ where $\{i_1, i_2 \ldots i_k\} \subseteq \{1, 2 \ldots n\}$

these form a basis $\mathcal{T}^k(V)$

Therefore $\dim \mathcal{T}^k(V) = n^k$

Proof: clearly $\varphi_{i_1}\otimes\varphi_{i_2}\otimes\dots\otimes\varphi_{i_k}\in\mathcal{T}^k(V)$

since $\varphi_{i_j}\in V^* = \mathcal{T}^1(V)$

The set spans $\mathcal{T}^k(V)$ and is linearly independent

Let $T\in\mathcal{T}^k(V)$. Need to write

$$T = \sum_{\substack{i_1=1\dots n\\ i_2=1\dots n\\ i_3=1\dots n\\ \vdots\\ i_k=1\dots n}} a^{i_1,i_2,\dots,i_n}\,\varphi_{i_1}\otimes\varphi_{i_2}\otimes\dots\otimes\varphi_{i_n}$$

Plug $(v_{j_1},v_{j_2},\dots,v_{j_k})$ into suspected identity.

$$T(v_{j_1},\dots,v_{j_k}) = \sum_{i_1,\dots,i_k} a^{i_1\dots i_k}\,\varphi_{i_1}\otimes\varphi_{i_2}\otimes\dots\otimes\varphi_{i_k}(v_{j_1},v_{j_2},\dots,v_{j_k})$$

$$= \sum_{i_1\dots i_k} a^{i_1\dots i_k}\,\varphi_{i_1}(v_{j_1})\varphi_{i_2}(v_{j_2})\dots\varphi_{i_k}(v_{j_k})$$

$$= \sum_{i_1\dots i_k} a^{i_1\dots i_k}\,\delta_{i_1j_1}\delta_{i_2j_2}\dots\delta_{i_kj_k}$$

$$= a^{j_1\dots j_k}$$

Define $a^{j_1\dots j_k} = T(v_{j_1},v_{j_2},\dots,v_{j_k})$

Let $w_1\dots w_k\in V$

$$w_1 = \sum_{j=1}^n a^{1j}v_j$$

$$w_2 = \sum_{j=1}^n a^{2j}v_j$$

$\vdots$

$$w_k = \sum_{j=1}^n a^{kj}v_j$$

$$T(w_1\dots w_k) = T\Big(\sum_{j=1}^n a^{1j}v_j,\ \sum_{j=1}^n a^{2j}v_j,\ \dots,\ \sum_{j=1}^n a^{kj}v_j\Big)$$

$$= T\Big(\sum_{j_1} a^{1j_1}v_{j_1},\ \dots,\ \sum_{j_k} a^{kj_k}v_{j_k}\Big)$$

T, k multilinear $$= \sum_{j_1,j_2\dots j_k=1}^n a^{1j_1}a^{2j_2}\dots a^{kj_k}\cdot T(v_{j_1},v_{j_2},\dots,v_{j_k})$$

$$= \sum_{j_1\dots j_k=1}^n a^{1j_1}\dots a^{kj_k}\,a^{j_1\dots j_k}$$

$$\cancel{\sum_{i_1\dots i_k} a^{i_1\dots i_k}\,\varphi_{i_1}\otimes\varphi_{i_2}\otimes\dots\otimes\varphi_{i_k}}$$

$$\sum_{i_1 \dots i_k} a^{i_1 \dots i_k} \varphi_{i_1} \otimes \varphi_{i_2} \otimes \dots \otimes \varphi_{i_k} (\omega_1, \omega_2 \dots \omega_k)$$

$$= \sum_{i_1 \dots i_k} a^{i_1 \dots i_k} \varphi_{i_1}(\omega_1) \otimes \varphi_{i_2}(\omega_2) \otimes \dots \otimes \varphi_{i_k}(\omega_k)$$

$$= \sum_{i_1 \dots i_k} a^{i_1 \dots i_k} a^{1 i_1} a^{2 i_2} \dots a^{k i_k} \quad \text{, relabel } i_1 \to j_1 \text{, } i_2 \to j_2 \text{ etc.}$$

$\varphi_{i_1} \otimes \dots \otimes \varphi_{i_k}$ are LI

$$\sum_{i_1 \dots i_k = 1}^{n} a^{i_1 i_2 \dots i_k} \varphi_{i_1} \otimes \varphi_{i_2} \otimes \dots \otimes \varphi_{i_k} = 0$$

Plug $v_{j_1}, v_{j_2}, \dots v_{j_k}$

$$\sum_{i_1 \dots i_k} a^{i_1 \dots i_k} \varphi_{i_1} \otimes \varphi_{i_2} \otimes \dots \otimes \varphi_{i_k} (v_{j_1}, v_{j_2} \dots v_{j_k})$$

$$= \sum_{i_1 \dots i_k} a^{i_1 \dots i_k} \delta_{i_1 j_1} \delta_{i_2 j_2} \dots \delta_{i_k j_k}$$

$$= a^{j_1 \dots j_k} = 0.$$

f even $f(-x) = f(x)$

f odd $f(-x) = -f(x)$

Every function $f : \mathbb{R} \to \mathbb{R}$ can be written as $f = \underset{\text{even}}{f_1} + \underset{\text{odd}}{f_2}$

$$f_1(x) = \frac{f(x) + f(-x)}{2} \qquad f_2(x) = \frac{f(x) - f(-x)}{2}.$$

$\sigma : X \to X$

σ is bijection $\mathbb{R} \to \mathbb{R}$

$\sigma^2 = \mathrm{Id}$

$$\frac{f(x) + f(\sigma x)}{2}.$$

Let S_k be the symmetric group on k, letters

$S_k \longrightarrow \{\pm 1\}$ multiplicative group (homomorphism).

$$\sigma \longrightarrow \begin{cases} +1 & \text{if } \sigma\text{-even} \\ -1 & \text{if } \sigma\text{-odd} \end{cases}$$

$\sigma \longrightarrow \operatorname{sgn}(\sigma)$

Definition:

If $T \in J^k(V)$ we define

$$\operatorname{Alt}(T)(\omega_1, \omega_2, \dots \omega_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\, T(\omega_{\sigma(1)}, \omega_{\sigma(2)}, \dots \omega_{\sigma(k)})$$

eg. $k=2$ $\operatorname{Alt}(T)(\omega_1, \omega_2) = \frac{1}{2!}\left(T(\omega_1, \omega_2) - T(\omega_2, \omega_1)\right)$

Theorem:

a). If $T \in \mathcal{T}^k(V)$, $\operatorname{Alt}(T) \in \mathcal{T}^k(V)$ and $\operatorname{Alt}(T)$ is alternating

b) If ω is alternating, $\operatorname{Alt}(\omega) = \omega$

c) $\operatorname{Alt}(\operatorname{Alt}(T)) = \operatorname{Alt}(T)$

Definition:

The set of alternating k-tensors is denoted by $\Lambda^k(V)$. It is a subspace of $\mathcal{T}^k(V)$

Proof:

c. follows from b

Use $\omega = \operatorname{Alt}(T)$ which by a is alternating

$\operatorname{Alt}(T) = \omega \stackrel{b}{=} \operatorname{Alt}(\omega) = \operatorname{Alt}(\operatorname{Alt}(T))$

a) Show $Alt(T) \in T^k(V)$ Ex

I will show it is alternating

$Alt(T)(\omega_1, \dots \omega_i, \dots \omega_j, \dots \omega_k) = -Alt(T)(\omega_1, \dots \omega_j, \dots \omega_i, \dots \omega_k)$

alternating

$i \to j$

$j \to i$ if $k \neq i,j$ $k \to k$

(i, j)

$S_k \to S_k$ bijection

$\sigma \to \sigma(i,j) = \sigma'$

even $\to$ odd

odd $\to$ even

$\sigma_1 \to \sigma_1(i,j) \Rightarrow \sigma_1 = \sigma_2$

$\sigma_2 \to \sigma(i,j)$

$Alt(T)(\omega_1, \dots \omega_j, \dots \omega_i \dots \omega_k)$

$= \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) T(\omega_{\sigma(1)} \dots \omega_{\sigma(j)} \dots \omega_{\sigma(i)} \dots \omega_{\sigma(k)})$

$= \frac{1}{k!} \sum_{\sigma' \in S} -sgn(\sigma') T(\omega_{\sigma'(1)}, \dots \omega_{\sigma'(i)}, \dots \omega_{\sigma'(j)} \dots \omega_{\sigma'(k)})$

$= -\frac{1}{k!} \sum_{\sigma' \in S} sgn(\sigma') T(\omega_{\sigma'(1)}, \dots \omega_{\sigma'(k)}) = -Alt(T)(\omega_1 \dots \omega_k)$

b) Let ϖ be alternating

$\varpi(\omega_1, \dots, \omega_j, \dots \omega_i, \dots \omega_k) = -\varpi(\omega_1, \dots \omega_i, \dots \omega_j, \dots \omega_k$

$\varpi(w_{\sigma(1)}, w_{\sigma(2)}, \dots w_{\sigma(k)}) = sgn(\sigma) \varpi(w_1, w_2 \dots w_k)$

$\sigma \in S_k$.

$Alt(\varpi)(w_1 \dots w_k) = \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) \varpi(w_{\sigma(1)}, \dots, w_{\sigma(k)})$

$= \frac{1}{k!} \sum_{\sigma \in S_k} \cancel{sgn(\sigma)} \cancel{sgn(\sigma)} \varpi(w_1 \dots w_k)$

$= \frac{1}{k!} \frac{|S_k|}{k!} \varpi(w_1, w_2, \dots, w_k) \Rightarrow Alt(\varpi) = \varpi$

Remark: If $\varpi \in \Lambda^k(V)$, $\eta \in \Lambda^l(V)$, then $\varpi \otimes \eta \in \mathcal{T}^{k+l}(V)$

~~Define $\varpi \wedge \eta = \frac{(k+l)!}{k!\,l!} \mathrm{Alt}(\varpi \otimes \eta) \in \Lambda^{k+l}(V)$~~

Definition:

Define $\varpi \wedge \eta = \dfrac{(k+l)!}{k!\,l!} \mathrm{Alt}(\varpi \otimes \eta) \in \Lambda^{k+l}(V)$

Properties:

1. $(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta$
2. $\omega \wedge (\eta_1 + \eta_2) = \omega \wedge \eta_1 + \omega \wedge \eta_2$
3. $(a\omega) \wedge \eta = a(\omega \wedge \eta) = \omega \wedge (a\eta)$ $a \in \mathbb{R}$

   $\omega_1, \omega_2, \omega \in \Lambda^k(V)$, $\eta_1, \eta_2, \eta \in \Lambda^l(V)$
4. $\omega \wedge \eta = (-1)^{k \cdot l} \eta \wedge \omega$.

Let V, W be vector spaces $f: V \to W$ linear transformation
If T is a linear ~~transformation~~ functional on W, $T: W \to \mathbb{R}$
then $T \circ f$ is a linear functional on V.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[dr, "T\circ f"'] & W \arrow[d, "T"] \\ & \mathbb{R} \end{tikzcd}$$

Notation/Definition

$f^*(T) = T \circ f$
$f^*(T)$ is called the pull back of T by f.

$f^*: W^* \to V^*$ by $f^*(T) = T \circ f$.

Pullback of tensors
If T is a k-tensor on W ie $T \in \mathcal{J}^k(W)$, we define the pullback $f^*(T) \in \mathcal{J}^k(V)$ by
$v_i \in V$
$f^*(T)(v_1, v_2, \ldots v_k) = T(f(v_1), f(v_2) \ldots f(v_k))$
This is a k-tensor on V.
Need to show linearity in i-entry
Let $v_i, v_i' \in V$, $\lambda \in \mathbb{R}$.

$f^*(T)(v_1, v_2, \ldots \lambda v_i + v_i', v_{i+1}, \ldots v_k)$
$= T(f(v_1) \ldots f(\lambda v_i + v_i'), f(v_{i+1}), \ldots f(v_k))$
$\overset{f \text{ linear}}{=} T(f(v_1) \ldots \lambda f(v_i) + f(v_i'), \ldots f(v_k))$
$\overset{T \text{ linear}}{=} \lambda T(f(v_1) \ldots f(v_i) \ldots f(v_k)) + T(f(v_1) \ldots f(v_i') \ldots f(v_k))$
$= \lambda f^*(T)(v_1 \ldots v_i, \ldots v_k) + f^*(T)(v_1 \ldots v_i', \ldots v_k)$

Properties

a $f^*(T \otimes S) = f^*(T) \otimes f^*(S)$
$T \in \mathcal{J}^k(W)$, $S \in \mathcal{J}^l(W)$

b $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$
for $\omega \in \Lambda^k(W)$, $\eta \in \Lambda^l(W)$

If $T \in \mathcal{J}^k(V)$

$\text{Alt}(T)(w_1 \dots w_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) T(w_{\sigma(1)}, w_{\sigma(2)}, \dots, w_{\sigma(k)})$

We have seen a basis of $\mathcal{J}^k(V)$
consists of $\varphi_{i_1} \otimes \varphi_{i_2} \otimes \varphi_{i_3} \otimes \dots \otimes \varphi_{i_k}$
with $\{\varphi_i\}$ dual basis of $\{v_i\}$
$i_1, i_2 \dots i_k \in \{1, 2 \dots n\}$ $n = \dim V$

$\Lambda^k(V)$ is a subspace of $\mathcal{J}^k(V)$.

Our difficulty $(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta)$.

Theorem:

a. $S \in \mathcal{J}^k(V)$, $T \in \mathcal{J}^l(V)$ and $\text{Alt}(S) = 0$ then:
$\text{Alt}(S \otimes T) = \text{Alt}(T \otimes S) = 0$.

b. $\text{Alt}(\text{Alt}(\omega \otimes \eta) \otimes \theta) = \text{Alt}(\omega \otimes \eta \otimes \theta) = \text{Alt}(\omega \otimes \text{Alt}(\eta \otimes \theta))$
$\omega \in \Lambda^k(V)$ $\eta \in \Lambda^l(V)$, $\theta \in \Lambda^m(V)$

c. $(\omega \wedge \eta) \wedge \theta = \omega \wedge (\theta \wedge \eta)$
$= \frac{(k+l+m)!}{k!\,l!\,m!} \text{Alt}(\omega \otimes \eta \otimes \theta)$

Proof:

a) $\text{Alt}(S \otimes T) = (w_1 \dots w_{k+l})$ $S \otimes T \in \mathcal{J}^{k+l}(V)$
$= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma)(S \otimes T)(w_{\sigma(1)} \dots w_{\sigma(k)}, w_{\sigma(k+1)} \dots w_{\sigma(k+l)})$

Let G be the subgroup of S_{k+l}
$G = \{\sigma \in S_{k+l}, \sigma(k+1) = k+1, \sigma(k+2) = k+2, \dots \sigma(k+l) = k+l\}$
The contribution of these to the sum is
$\frac{1}{(k+l)!} \left[\sum_{\sigma \in G} \text{sgn}(\sigma) S(w_{\sigma(1)} \dots w_{\sigma(k)}) \right] \cdot T(w_{k+1} \dots w_{k+l})$
$= \frac{1}{(k+l)!} k!\, \text{Alt}(S)(w_1 \dots w_k) \cdot T(w_{k+1} \dots w_{k+l}) = 0$

Let $G\sigma_0$ be a coset of G in S_{k+l}

$\sigma_0 \neq id$ $\quad G\sigma_0 = \{\sigma' \cdot \sigma_0 \in G\}$

Define $(z_1, \dots z_{k+l}) = (w_{\sigma_0(1)}, \dots w_{\sigma_0(k+l)})$

The contribution of these elements is:

$$\frac{1}{(k+l)!} \sum_{\sigma' \in G} \text{sgn}(\sigma' \cdot \sigma_0)\, S(z_{\sigma'(1)}, \dots z_{\sigma'(k)}) \cdot T(z_{\sigma'(k+1)} \dots z_{\sigma'(k+l)})$$

but $\sigma' \in G$ so $\sigma'(k+1) = k+1$, $\sigma'(k+2) = k+2 \dots$

sgn homomorphism

$$\frac{1}{(k+l)!} \sum_{\sigma \in G} \text{sgn}(\sigma')\, \text{sgn}(\sigma_0)\, S(z_{\sigma'(1)}, \dots z_{\sigma'(k)}) \cdot T(z_{k+1} \dots z_{k+l})$$

$$\frac{1}{(k+l)!} \text{sgn}(\sigma_0) T(z_{k+1} \dots z_{k+l})\, k!\, \text{Alt}\, S(z_{\sigma'(1)} \dots z_{\sigma'(k)}) = 0$$

b) $\text{Alt}(\omega \otimes \eta) - \omega \otimes \eta = S$

$\text{Alt}(S) = \text{Alt}(\text{Alt}(\omega \otimes \eta) - \omega \otimes \eta)$

$\quad = \text{Alt}(\text{Alt}(\omega \otimes \eta)) - \text{Alt}(\omega \otimes \eta)$

$\quad = \text{Alt}(\omega \otimes \eta) - \text{Alt}(\omega \otimes \eta) = 0$

Apply a. with this S

$\text{Alt}(S \otimes \theta) = 0$

$\text{Alt}([\text{Alt}(\omega \otimes \eta) - \omega \otimes \eta] \otimes \theta) = 0$

$\text{Alt}(\text{Alt}(\omega \otimes \eta) \otimes \theta) - \text{Alt}(\omega \otimes \eta \otimes \theta) = 0$

c) $$(\omega \wedge \eta) \wedge \theta = \frac{(k+l+m)!}{(k+l)!\, m!} \text{Alt}((\omega \wedge \eta) \otimes \theta)$$

$$= \frac{(k+l+m)!}{(k+l)!\, m!} \text{Alt}\left(\frac{(k+l)!}{k!\, l!} \text{Alt}(\omega \otimes \eta) \otimes \theta \right)$$

$$= \frac{(k+l+m)!\, \cancel{(k+l)!}}{\cancel{(k+l)!}\, k!\, l!\, m!} \text{Alt}(\text{Alt}(\omega \times \eta) \otimes \theta)$$

$$= \frac{(k+l+m)!}{k!\, l!\, m!} \text{Alt}(\omega \otimes \eta \otimes \theta).$$

Theorem:

Let $\dim V = n$ then the following is a basis of $\Lambda^k(V)$
$\varphi_{i_1} \wedge \varphi_{i_2} \wedge \dots \wedge \varphi_{i_k} \qquad 1 \le i_1 < i_2 < \dots < i_k \le n$
Therefore $\dim \Lambda^k(V) = \binom{n}{k}$ (take subset of size k from $\{1, \dots n\}$ and order them increasingly).

Corollary:

$k > n \quad \Lambda^k(V) = \{0\}$
$k = 1 \quad \dim \Lambda^k(V) = \binom{n}{1} = n$
Since Λ^1 has tensors with 1 slot $\Lambda^1(V) = \mathcal{J}^1(V) = V^*$
$\dim \Lambda^n(V) = \binom{n}{n} = 1$
$\det(v_1, \dots v_n) \in \Lambda^n(V)$ and since $\det(I) = 1$ every alternating tensor is a multiple of $\det(v_1 \dots v_n)$.

Proof of theorem: $T \in \Lambda^k(V)$ then $\mathrm{Alt}(T) = T$
Since $\varphi_{i_1} \otimes \varphi_{i_2} \otimes \dots \otimes \varphi_{i_k}$ is basis of $\mathcal{J}^k(V)$
$$T = \sum_{i_1 \dots i_k = 1}^{n} a^{i_1 i_2 \dots i_k} \varphi_{i_1} \otimes \varphi_{i_2} \otimes \dots \otimes \varphi_{i_k}$$

Apply Alt on both sides
$$T = \sum_{i_1 \dots i_k}^{n} a^{i_1 i_2 \dots i_k} \mathrm{Alt}(\varphi_{i_1} \otimes \varphi_{i_2} \otimes \dots \otimes \varphi_{i_k})$$

$\mathrm{Alt}(\varphi_{i_1} \otimes \dots \otimes \varphi_{i_k})$ is a multiple of $\varphi_{i_1} \wedge \varphi_{i_2} \dots \wedge \varphi_{i_k}$
since $\varphi_{i_j} \wedge \varphi_{i_s} = -\varphi_{i_s} \wedge \varphi_{i_j}$
you can render to $\sum_{i_1 < i_2 < \dots < i_k}$

So $\varphi_{i_1} \wedge \varphi_{i_2} \wedge \dots \wedge \varphi_{i_k}$ with $i_1 < i_2 < \dots < i_k$ span generate $\Lambda^k(V)$
It is easy to see that they are LI.

Example

dim $V=3$

$K=1$ $\dim \Lambda^1(V) = \binom{3}{1} = 3$

$\Lambda^1(V) = \mathcal{J}^1(V) = V^*$

If $\{v_i\}_{i=1,2,3}$ is basis of V then the dual basis $\varphi_1, \varphi_2, \varphi_3$ is basis of $\Lambda^1(V)$

$K=2$ $\dim \Lambda^2(V) = \binom{3}{2} = 3$

Basis is $\varphi_1 \wedge \varphi_2$ $\varphi_1 \wedge \varphi_3$ and $\varphi_2 \wedge \varphi_3$.

$$(\varphi_1 \wedge \varphi_2)(w_1, w_2) = \frac{(1+1)!}{1!1!} \operatorname{Alt}(\varphi_1 \otimes \varphi_2)(w_1, w_2)$$

$$= 2! \frac{1}{2!} \left(\varphi_1 \otimes \varphi_2 (w_1, w_2) - \varphi_1 \otimes \varphi_2 (w_2, w_1) \right)$$

$$= \varphi_1(w_1)\varphi_2(w_2) - \varphi_1(w_2)\varphi_2(w_1)$$

$$= \varphi_1(w_1)\varphi_2(w_2) - \varphi_2(w_1)\varphi_1(w_2)$$

$$= (\varphi_1 \otimes \varphi_2)(w_1, w_2) - (\varphi_2 \otimes \varphi_1)(w_1, w_2)$$

$$\boxed{\begin{aligned} \varphi_1 \wedge \varphi_2 &= \varphi_1 \otimes \varphi_2 - \varphi_2 \otimes \varphi_1 \\ \varphi_1 \wedge \varphi_3 &= \varphi_1 \otimes \varphi_3 - \varphi_3 \otimes \varphi_1 \\ \varphi_2 \wedge \varphi_3 &= \varphi_2 \otimes \varphi_3 - \varphi_3 \otimes \varphi_2 \end{aligned}}$$

$$\boxed{\begin{aligned} \varphi_2 \wedge \varphi_1 &= \varphi_2 \otimes \varphi_1 - \varphi_1 \otimes \varphi_2 = -\varphi_1 \wedge \varphi_2 \\ \varphi_3 \wedge \varphi_1 &= -\varphi_1 \wedge \varphi_3 \\ \varphi_3 \wedge \varphi_2 &= -\varphi_3 \wedge \varphi_2. \end{aligned}}$$

$$(\varphi_1 \wedge \varphi_1)(w_1, w_2) = \varphi_1(w_1)\varphi_1(w_2) - \varphi_1(w_1)\varphi_1(w_2) = 0$$

$$\boxed{\varphi_1 \wedge \varphi_1 = 0 \quad \varphi_2 \wedge \varphi_2 = 0 \quad \varphi_3 \wedge \varphi_3}.$$

$K=3$ $\dim \Lambda^3(V) = \binom{3}{3} = 1$

basis $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$

$$(\varphi_1 \wedge \varphi_2 \wedge \varphi_3)(w_1, w_2, w_3) = 3! \operatorname{Alt}(\varphi_1 \otimes \varphi_2 \otimes \varphi_3)(w_1, w_2, w_3)$$

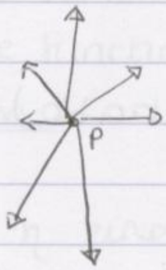
$$= \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma)(\varphi_1 \otimes \varphi_2 \otimes \varphi_3)(w_{\sigma(1)}, w_{\sigma(2)}, w_{\sigma(3)}).$$

$$= \varphi_1(w_1)\varphi_2(w_2)\varphi_3(w_3) - \varphi_1(w_2)\varphi_2(w_1)\varphi_3(w_3) - \varphi_1(w_3)\varphi_2(w_2)\varphi_3(w_1)$$
$$- \varphi_1(w_1)\varphi_2(w_3)\varphi_3(w_2) + \varphi_1(w_2)\varphi_2(w_3)\varphi_1(w_1) + \varphi_1(w_3)\varphi_2(w_1)\varphi_3(w_2)$$

$$\varphi_1 \wedge \varphi_2 \wedge \varphi_3 = \varphi_1 \otimes \varphi_2 \otimes \varphi_3 - \varphi_2 \otimes \varphi_1 \otimes \varphi_3 - \varphi_3 \otimes \varphi_2 \otimes \varphi_1 - \varphi_1 \otimes \varphi_3 \otimes \varphi_2 + \varphi_3 \otimes \varphi_1 \otimes \varphi_2 + \varphi_2 \otimes \varphi_3 \otimes \varphi_1$$

Consider

$\mathbb{R}^n$

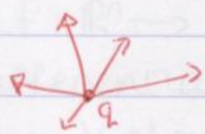


$\mathbb{R}^n_p = \{(p,v), v \in \mathbb{R}^n\}$

This is the tangent space at p.

$(p,v) + (p,w) = (p, v+w)$

$\lambda(p,v) = (p, \lambda v)$ with these operations $\mathbb{R}^n_p$ is a vector space.



If $p \neq q$ it makes no sense to consider $\cancel{(p,v) + (q,w)}$

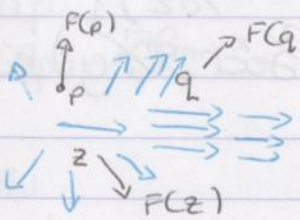
Notation: $v_p = (p,v)$

On $\mathbb{R}^n_p$ $\langle (p,v), (p,w) \rangle = \langle v, w \rangle$

Definition:

A vector field in $\mathbb{R}^n$ is a function $p \xrightarrow{F} F(p) \in \mathbb{R}^n_p$

$F(p) = (p, v)$



$v = (F^1(p), F^2(p) \dots F^n(p))$

$p \rightarrow F^i(p)$

If the components F^i $i \in \{1, \dots, n\}$ are continuous the vector field is continuous

If the components are differentiable, the vector field is differentiable.

If F, G are vector fields in $\mathbb{R}^n$, $F+G$ also a vector field in $\mathbb{R}^n$

$(F+G)(p) = F(p) + G(p)$

$\lambda \in \mathbb{R}$ $\lambda \cdot F$ is vector field $(\lambda F)(p) = \lambda \cdot F(p)$

If $f : \mathbb{R}^n \to \mathbb{R}$ is a function (continuous, differentiable) then $f \cdot F$ is a new vector field on $\mathbb{R}^n$

$(f \cdot F)(p) = f(p) \cdot F(p)$

Definition:

If F is a vector field then its divergence is

$$(\operatorname{div} F)\cdot(p) = \sum_{i=1}^{n} \partial_i F^i(p) \in \mathbb{R}$$

So $\operatorname{div} F : \mathbb{R}^n \to \mathbb{R}$.

Notation: $\operatorname{div} F = \nabla \cdot F$.

Definition:

Also in $F : \mathbb{R}^3 \to \mathbb{R}$ you have seen rotation or curling of the vector field defined by

$$\nabla \times F = \operatorname{curl}(F) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F^1 & F^2 & F^3 \end{vmatrix} = \left(\partial F^3/\partial y - \partial F^2/\partial z\right) i - \left(\partial F^3/\partial x - \partial F^1/\partial z\right) j + \left(\partial F^2/\partial x - \partial F^1/\partial y\right) \underline{k}$$

$$= \operatorname{rot}(F)$$

Given $p \in \mathbb{R}^n$ let $\omega(p) \in \Lambda^k(\mathbb{R}^n_p)$

$$\omega(p) = \sum_{i_1 < i_2 < \dots < i_k} \omega_{i_1, i_2 \dots i_k}(p)\, \varphi_{i_1}(p) \wedge \varphi_{i_2}(p) \dots \wedge \varphi_{i_k}(p)$$

This is a <u>k-form</u> on $\mathbb{R}^n$

It is determined by $\binom{n}{k}$ functions $p \longrightarrow \omega_{i_1, i_2 \dots i_k}(p)$ $\quad i_1 < i_2 < \dots < i_k$

If these functions are continuous then the <u>k-form is continuous</u>

If these functions are differentiable then <u>ω is a differential k-form</u>.

If ω, η are differentiable k-forms on $\mathbb{R}^n$, $\omega + \eta$ is a differentiable k-form on $\mathbb{R}^n$

$$(\omega + \eta)(p) = \underbrace{\omega(p)}_{\in \Lambda^k(\mathbb{R}^n_p)} + \underbrace{\eta(p)}_{\in \Lambda^k(\mathbb{R}^n_p)}$$

If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a (differentiable) function then $f \cdot \omega$ is a differentiable k-form

$$(f \cdot \omega)(p) = \underbrace{f(p)\, \omega(p)}_{\in \Lambda^k(\mathbb{R}^n_p)}$$

If ω is a differential k-form and η is a differential l-form then $\omega \wedge \eta$ is a differential $k+l$-form.

$$(\omega \wedge \eta)(p) = \underbrace{\omega(p)}_{\in \Lambda^k(\mathbb{R}^n_p)} \wedge \underbrace{\eta(p)}_{\in \Lambda^l(\mathbb{R}^n_p)}$$

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable then $Df(p) : \mathbb{R}^n \longrightarrow \mathbb{R}$ linear map.

$$Df(p) \in (\mathbb{R}^n_p)^* = \mathcal{J}^1(\mathbb{R}^n_p) = \Lambda^1(\mathbb{R}^n_p)$$

Beku

Definition:

We define df to be the following 1-form
$df(p) \in \Lambda^1(\mathbb{R}^n_p)$
$df(p)(v_p) = Df(p)(v)$

Example

Let $f = \pi^i$ the projection into i-component
$\pi^i(x^1, x^2 \dots x^n) = x^i$ linear map
Sometimes it is denoted $x^i(x) = x^i$
$d\pi^i(p)(v_p) = D\pi^i(p)(v)$ $\quad v = (v^1, v^2 \dots v^n)$
$= \pi^i(p)(v) = \pi^i(v) = v^i$
$= \varphi_i(v)$

$d\pi^i = \varphi_i = dx^i$

A differential form will look like
$$\omega(p) = \sum_{i_1 < \dots < i_k} \omega_{i_1 i_2 \dots i_k}(p)\, dx^{i_1}(p) \wedge dx^{i_2}(p) \wedge \dots \wedge dx^{i_k}(p)$$

$$\omega = \sum_{i_1 < i_2 < \dots < i_k} \omega_{i_1 i_2 \dots i_k}\, dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k}$$

Example

$\mathbb{R}^3$ $(x^1, x^2, x^3) = (x, y, z)$

$K=1$ $\quad \omega = f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz.$

$K=2$ $\quad \omega = f(x,y,z)\,dx \wedge dy + g(x,y,z)\,dx \wedge dz$
$+ h(x,y,z)\,dy \wedge dx.$

$K=3$ $\quad \omega = f(x,y,z)\,dx \wedge dy \wedge dz$

$k=0$ & $\omega = f(x,y,z)$

$dx \wedge dx = 0 \qquad dx \wedge dy = -dy \wedge dx$

$dy \wedge dy = 0 \qquad dx \wedge dy = -dy \wedge dx$

$dz \wedge dz = 0 \qquad dy \wedge dz = -dz \wedge dy$

Example.

$n=2$

$k=1 \quad \omega = f(x,y)dx + g(x,y)dy$

$k=2 \quad \omega = f(x,y)dx \wedge dy.$

Theorem:

Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable then the one form df is

$df = D_1 f\, dx^1 + D_2 f\, dx^2 + \dots + D_n f\, dx^n$

(eg in $\mathbb{R}^3$ $\quad df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$)

Proof: $df(p) \in \Lambda^1(\mathbb{R}^n_p)$

$$\begin{aligned} df(p)(v_p) &\overset{\text{def}}{=} Df(p)(v) \\ &= (D_1 f(p), \dots D_n f(p)) \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} \\ &= \sum_{i=1}^{n} D_i f(p) v^i \end{aligned}$$

Calculate $(D_1 f\, dx^1 + \dots + D_n f\, dx^n)(p)(v_p)$

$$\begin{aligned} &= [D_1 f(p) dx^1(p) + \dots + D_n f(p) dx^n(p)](v_p) \\ &= D_1 f(p) dx^1(p)(v_p) + \dots + D_n f(p) dx^n(p)(v_p) \\ &= D_1 f(p) \cdot v^1 + \dots + D_n f(p) \cdot v^n \end{aligned}$$

Definition:

~~The operator~~

The operator $\cancel{d}$ on K-forms

$K=0 \quad \omega=f \quad df=\sum_i D_i f\, dx^i$ 1-form

In general $\underset{K\text{-form}}{\omega}=\sum_{i_1<i_2<\dots<i_k} \omega_{i_1 i_2 \dots i_k}\, dx^{i_1}\wedge dx^{i_2}\wedge\dots\wedge dx^{i_k}$

Define $d\omega=\sum_{i_1<i_2<\dots<i_k}\ \sum_{i=1}^{n} D_i\,\omega_{i_1 i_2\dots i_k}\, dx^i\wedge dx^{i_1}\wedge dx^{i_2}\wedge\dots\wedge dx^{i_k}$

K+1 form.

~~Ex~~ Example:

$n=3 \quad K=1 \quad \omega=f\,dx+g\,dy+h\,dz$

$$d\omega=\frac{\partial f}{\partial x}\cancel{dx\wedge dx}+\frac{\partial f}{\partial y}dy\wedge dx+\frac{\partial f}{\partial z}dz\wedge dx$$
$$+\frac{\partial g}{\partial x}dx\wedge dy+\frac{\partial g}{\partial y}\cancel{dy\wedge dy}+\frac{\partial g}{\partial z}dz\wedge dy$$
$$+\frac{\partial h}{\partial x}dx\wedge dz+\frac{\partial h}{\partial y}dy\wedge dz+\frac{\partial h}{\partial z}\cancel{dz\wedge dz}$$
$$=-\frac{\partial f}{\partial y}dx\wedge dy-\frac{\partial f}{\partial z}dx\wedge dz+\frac{\partial g}{\partial x}dx\wedge dy$$
$$-\frac{\partial g}{\partial z}dy\wedge dz+\frac{\partial h}{\partial x}dx\wedge dz+\frac{\partial h}{\partial y}dy\wedge dz.$$
$$=\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)dx\wedge dy+\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)dy\wedge dz$$
$$\overset{-}{\cancel{+}}\left(\frac{\partial h}{\partial x}-\frac{\partial f}{\partial z}\right)d\underset{z}{x}\wedge d\underset{x}{z}.$$

Consider
$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f & g & h \end{vmatrix} = \begin{matrix}(\partial h/\partial y-\partial g/\partial z)\,\underline{i} \\ -(\partial h/\partial x-\partial f/\partial z)\,\underline{j} \\ +(\partial g/\partial x-\partial f/\partial z)\,\underline{k}\end{matrix}$$

$\underline{i}\longleftrightarrow dy\wedge dz$

$\underline{j}\longleftrightarrow dz\wedge dx$

$k\longleftrightarrow dx\wedge dy$

Example

$n=3$

$k=2 \quad \omega = f_1\, dy\wedge dz + f_2\, dz\wedge dx + f_3\, dx\wedge dy$

$$d\omega = \frac{\partial f_1}{\partial x} dx\wedge dy\wedge dz + \frac{\partial f_1}{\partial y} dy\wedge \cancel{dy}\wedge dz + \frac{\partial f_2}{\partial z} dz\wedge \cancel{dy}\wedge dz$$

$$+ \frac{\partial f_2}{\partial y} dy\wedge dz\wedge dx \quad + \frac{\partial f_3}{\partial z} dz\wedge dx\wedge dy$$

$$= \frac{\partial f_1}{\partial x} dx\wedge dy\wedge dz + \frac{\partial f_2}{\partial y} dx\wedge dy\wedge dz$$

$$+ \frac{\partial f_3}{\partial z} dx\wedge dy\wedge dz.$$

$$= \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial x}\right) dx\wedge dy\wedge dz.$$

$\omega \longleftrightarrow F = (f_1, f_2, f_3)$

$$\operatorname{div} F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \longleftrightarrow d\omega.$$

Example

$n=3$

$k=3 \quad \omega = f(x,y,z)\, dx\wedge dy\wedge dz$

$d\omega = 0 \quad$ 4-form on $\mathbb{R}^3$ $\binom{3}{4} = 0.$

Example.

$n=2$

$k=1 \quad \omega = f(x,y)\,dx + g(x,y)\,dy$

$$d\omega = \frac{\partial f}{\partial x} \cancel{dx\wedge dx} + \frac{\partial f}{\partial y} dy\wedge dx + \frac{\partial g}{\partial x} dx\wedge dy$$

$$+ \frac{\partial g}{\partial y} \cancel{dy\wedge dy}$$

$$= \left(-\frac{\partial f}{\partial y} + \frac{\partial g}{\partial x}\right) dx \wedge dy$$

Greens Thm: $\int_\gamma f\,dx + g\,dy = \int_\Omega \left(-\frac{\partial f}{\partial y} + \frac{\partial g}{\partial x}\right) dx\,dy.$

Example

$k=2 \quad \omega = f(x,y)\,dx \wedge dy$

$d\omega = 0$

Example

$n=1 \quad k=1 \quad \omega = f(x)\,dx \quad d\omega = 0.$

Theorem:

a) $d(\omega + \eta) = d\omega + d\eta \quad \omega, \eta$ k-form.

b) ω is k-form, η is l-form ($\omega \wedge \eta$ k+l form)

$\underbrace{d(\omega \wedge \eta)}_{k+l+1 \text{ form}} = d(\omega) \wedge \eta + (-1)^k \omega \wedge d(\eta)$

c) $d(d(\omega)) = 0$

Proof

c) $\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k}\, dx^{i_1} \wedge \dots \wedge dx^{i_k}$

$$d(d\omega) = d\left(\sum_{i_1 < \dots < i_k} \sum_{i=1}^{n} D_i \omega_{i_1 \dots i_k}\, dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}\right)$$

$$= \sum_{i_1 < \dots < i_k} \sum_{i=1}^{n} \sum_{j=1}^{n} D_j (D_i \omega_{i_1 \dots i_k})\, dx^j \wedge dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

If $i=j$ $\quad dx^i \wedge dx^i = 0$ ignore.

$i \neq j$ (i,j) (j,i) $D_j D_i \omega_{i_1 \dots i_k} dx^j \wedge dx^i - D_i D_j \omega_{i_1 \dots i_k} dx^j \wedge dx^i$

For functions like $\omega_{i_1 \dots i_k}$ which have continuous mixed partial derivatives we have proved $D_i D_j \omega_{i_1 \dots i_k} = D_j D_i \omega_{i_1 \dots i_k}$.

b) $\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1, i_2, \dots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k}$

Take $\eta = \sum_{j_1 < j_2 < \dots < j_l} \eta_{j_1 \dots j_l} dx^{j_1} \wedge dx^{j_2} \wedge \dots \wedge dx^{j_l}$

$$\omega \wedge \eta = \sum_{i_1 < \dots < i_k} \sum_{j_1 < \dots < j_l} \omega_{i_1 \dots i_k} \eta_{j_1 \dots j_l} dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l}$$

$$d(\omega \wedge \eta) = \sum_{i_1 < \dots < i_k} \sum_{j_1 < \dots < j_l} \sum_{\alpha = 1}^{n} D_\alpha (\omega_{i_1 \dots i_k} \eta_{j_1 \dots j_l}) dx^\alpha \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l}$$

$$= \sum_{i_1 \dots i_k} \sum_{j_1 \dots j_l} \sum_{\alpha = 1}^{n} (D_\alpha \omega_{i_1 \dots i_k} \eta_{j_1 \dots j_l} + \omega_{i_1 \dots i_k} \cdot D_\alpha \eta_{j_1 \dots j_l}) dx^\alpha \wedge dx^{i_1} \wedge \dots \wedge dx^{j_l}$$

$$= \sum_{i_1 < \dots < i_k} \sum_{j_1 < \dots < j_l} \sum_{\alpha = 1}^{n} D_\alpha \omega_{i_1 \dots i_k} \cdot \eta_{j_1 \dots j_l} dx^\alpha \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l}$$

$$+ \sum_{i_1 < \dots < i_k} \sum_{j_1 < \dots < j_l} \sum_{\alpha = 1}^{n} \omega_{i_1 \dots i_k} \cdot D_\alpha \eta_{j_1 \dots j_l} dx^\alpha \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l}$$

$$= \Big(\sum_{i_1 < \dots < i_k} \sum_{\alpha = 1}^{n} D_\alpha \omega_{i_1 \dots i_k} dx^\alpha \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} \Big) \wedge \sum_{j_1 < \dots < j_l} \eta_{j_1 \dots j_l} dx^{j_1} \wedge \dots \wedge dx^{j_l}$$

$$+ \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge (-1)^k \sum_{j_1 < \dots < j_l} \sum_{\alpha = 1}^{n} D_\alpha \eta_{j_1 \dots j_l} dx^\alpha \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l}$$

$$= d\omega \wedge \eta + \omega \wedge (-1)^k d\eta.$$

Closed and exact forms.

Let ω be a k-form.

Definition:

ω is called ~~a~~ closed ~~form~~ if $d\omega = 0$
ω is called exact if $\exists$ ~~an~~ a $(k-1)$-form η st $d\eta = \omega$.

Proposition:

If ω is exact then it is closed

Proof exact $\Rightarrow \omega = d\eta$ then $d\omega = d(d\eta) = 0$.

Example

$n=2,\ k=1 \quad \omega = P(x,y)dx + Q(x,y)dy$

$$d\omega = \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right)dx \wedge dy$$

ω is closed if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$

vector field $F = Pi + Qj$

If $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ we call it conservative vector field

F is conservative if F has a potential function f.
$F = \text{grad}(f)$

When is ω exact?

0-form $\eta = f$

$$\omega = d\eta \Leftrightarrow \omega = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

ie $\omega = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy$

$$\text{grad}(f) = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j$$

Example

$\omega = xy^2 dx + y dy$

$d\omega = \dfrac{\partial(xy^2)}{\partial y} dy \wedge dx + \dfrac{\partial y}{\partial x} dx \wedge dy$

$= 2xy \, dy \wedge dx \neq 0$ not closed $\Rightarrow$ not exact.

Example

$\omega = xy^2 dx + x^2 y dx$

$d\omega = 2xy \, dy \wedge dx + 2xy \, dx \wedge dy$

$= -2xy \, dx \wedge dy + 2xy \, dx \wedge dy = 0$ closed

is it exact?

$\exists f = \dfrac{x^2y^2}{2} + K$

$df = \omega$

$\dfrac{\partial f}{\partial x} dx + \dfrac{\partial f}{\partial y} dy = \omega = x y^2 dx + x^2 y dy$

$\dfrac{\partial f}{\partial x} = xy^2 \qquad \dfrac{\partial f}{\partial y} = x^2 y$

$f(x,y) = \int xy^2 \, dx$

$= \dfrac{x^2y^2}{2} + c(y)$

$\dfrac{\partial f}{\partial y} = x^2 y + \dfrac{\partial c}{\partial y} = x^2 y \Rightarrow \dfrac{\partial c}{\partial y} = 0 \quad c = K$

$n > 2 \quad k = 1$

$\omega = \omega_1 dx^1 + \ldots + \omega_n dx^n$ is closed

Is it exact $\omega = df = D_1 f dx^1 + \ldots + d_n dx^n$

$\omega_i = D_i f$

I can assume $f(0) = 0$

You can recover f by an intergration in 1-variable t.

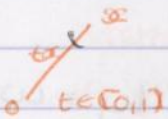
$$f(x) - f(0) = \int_0^1 \frac{d}{dt}[f(tx)]dx \qquad x \in \mathbb{R}^n$$

$$\overset{\text{F.T of calculus}}{=} f(tx)\Big|_{t=0}^{t=1}$$

$$f(x) = \int_0^1 \sum_{\alpha=1}^{n} D_\alpha f(tx) \frac{d}{dt}(tx^\alpha)\, dt$$

$$= \int_0^1 \sum_{\alpha=1}^{n} D_\alpha f(tx)\, x^\alpha\, dt$$

$$f(x) = \int_0^1 \sum_{\alpha=1}^{n} \omega_\alpha (tx) x^\alpha\, dt$$

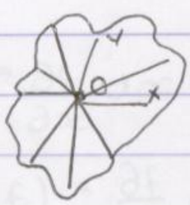


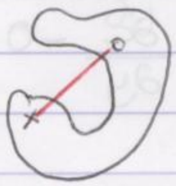
!!!

Definition:

A is a star-shaped region with respect to 0 (or p) if $\forall t \in [0,1]$, $\forall x \in A$ we $t \cdot x \in A$.

eg.





not a star shaped

Poincaré Lemma

If A is star-shaped w.r.t 0 and ω is a closed form on A then ω is an exact form on A.

Proof: For any l-form ω I will define ~~a~~ $(l-1)$-form $I(\omega)$ st $I(\lambda\omega_1 + \omega_2) = \lambda I(\omega_1) + I(\omega_2)$

$$I(0) = 0$$

and $\underbrace{d(\underbrace{I(\omega)}_{l-1})}_{l} + \underbrace{I(\underbrace{d\omega}_{l+1})}_{l} = \omega$

Then if ω is closed, $d\omega = 0$ so $I(d\omega) = 0$

So we get $d(I(\omega)) = \omega$

$\Rightarrow \omega$ is exact.

$$\omega = \sum_{i_1<\dots<i_l} \omega_{i_1\dots i_l}\, dx^{i_1}\wedge\dots\wedge dx^{i_l}$$

this is removed

$$I(\omega) = \sum_{i_1<\dots<i_l} \sum_{\alpha=1}^{l} (-1)^{\alpha-1} \int_0^1 t^{l-1}\omega_{i_1\dots i_l}(tx)\, x^{i_\alpha}\, dt\; dx^{i_1}\wedge dx^{i_2}\wedge\dots\wedge \widehat{dx^{i_\alpha}}\wedge\dots\wedge dx^{i_l}$$

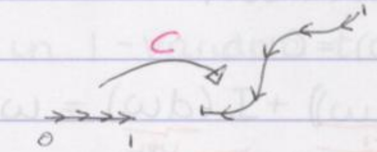
$$I(0) = 0$$

Geometric Preliminaries

Definition:

The set $I^k = [0,1]^k$ is called the standard k-cube.
A continuous map $c: I^k \to A$, where A is open in $\mathbb{R}^n$ is called a singular k-cube.

Eg. $k=1 \quad c:[0,1] \to A$ curve



$k=2 \quad c:[0,1]^2 \to A$ surface

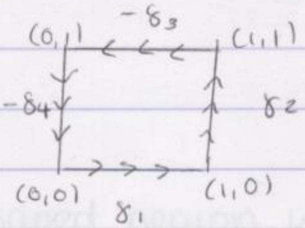
$k=0 \quad [0,1]^0 = \{0\}$. A singular 0-cube $\{0\} \to A$

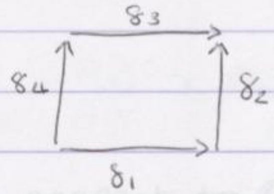
Consider $k=1$

$\partial(I^1) = +1 - 0$

$\int_0^1 f'(x)\,dx = f(1) - f(0)$.

$I^2 = [0,1]^2$





$\partial I^2 = \gamma_1 + \gamma_2 - \gamma_3 - \gamma_4$

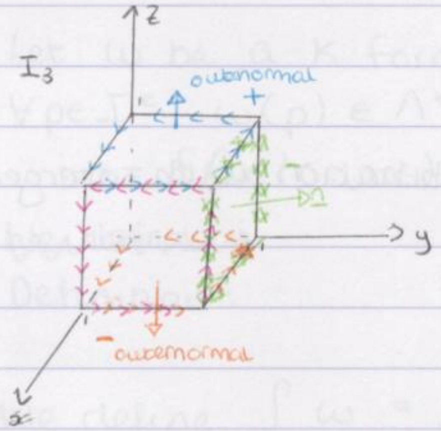
$\gamma_1 \cdot I^2_{(2,0)} = \{(x,0),\ 0 \le x \le 1\}$

$\gamma_2 \quad I^2_{(1,1)} = \{(1,y),\ 0 \le y \le 1\}$.

$\gamma_3 \quad I^2_{(2,1)} = \{(x,1),\ 0 \le x \le 1\}$.

$\gamma_4 \quad I^2_{(1,0)} = \{(0,y)\ 0 \le y \le 1\}$.

I_3



$$\partial I^3 = I^3_{(3,1)} - I^3_{(3,0)} + I^1_{(1,1)} - I^3_{(1,0)} + I^3_{(2,0)} - I^3_{(2,1)}$$

top $I^3_{(3,1)} = \{(x, y, 1), 0 \leq x \leq 1, 0 \leq y \leq 1\}$.
base $I^3_{(3,0)} = \{(x, y, 0)\ 0 \leq x \leq 1, 0 \leq y \leq 1\}$
front face $I^3_{(1,1)} = \{(1, y, z)\ 0 \leq y \leq 1, 0 \leq z \leq 1\}$.
back face $I^3_{(1,0)} = \{(0, y, z)\ 0 \leq y \leq 1, 0 \leq z \leq 1\}$.
left $I^3_{(2,0)} = \{(x, 0, z)\ 0 \leq x \leq 1, 0 \leq z \leq 1\}$.
right $I^3_{(2,1)} = \{(x, 1, z)\ 0 \leq x \leq 1, 0 \leq z \leq 1\}$.

Definition:

Given an n-cube $I^n = [0,1]^n$ we define the various faces of it to be

$I^n_{(i,0)} = \{(x^1, x^2, \ldots, x^{i-1}, 0, x^{i+1}, \ldots x^n), 0 \leq x^j \leq 1\}$.
$I^n_{(i,1)} = \{(x^1, x^2, \ldots x^{i-1}, 1, x^{i+1}, \ldots x^n), 0 \leq x^j \leq 1\}$.

We define the boundary of I^n to be

$$\partial I^n = \sum_{i=1}^{n} \sum_{\alpha=0}^{1} (-1)^{i+\alpha} I^n_{(i,\alpha)}$$

We form formal sums of singular n-cubes with interger coefficents (this is the construction of a certain abelian group or $\mathbb{Z}$ module).

eg $c_1 : I^1 \longrightarrow A$
$c_2 : I^1 \longrightarrow A$

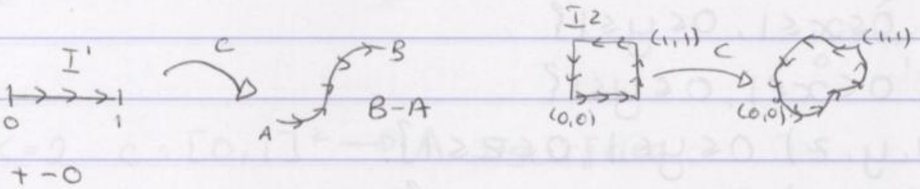
c_1 c_2

eg $3c_1 + (-5)c_2$, for example, is a singular n-chain.

Definition:

A singular n-chain c, is a (finite) linear combination with interger coefficents of singular n-cubes.

$$c = \sum_{j=1}^{m} m_j c_j \quad m \in \mathbb{Z} \quad c_j : I^n \to A.$$



Definition:

If c is a singular n-cube $(c : I^n \to A)$, then
$$\partial c = \sum_{i=1}^{n} \sum_{\alpha=0}^{1} (-1)^{i+\alpha} c(I^n_{(i,\alpha)})$$

For a singular n chain $c = \sum_{j=1}^{m} m_j c_j$ where c_j are singular n-cubes
$$\partial c = \sum_{j=1}^{n} m_j \partial(c_j)$$

Consider $\displaystyle\int_{\partial c} \omega = \int_c d\omega$

If ω is $k-1$ form $d\omega$ is a k form
c will be a singular k-chain. ∂c ~~which~~ is a singular $(k-1)$ chain.

Today on $\mathbb{R}^k$ we will define intergration of k form on a k-cube and $k-1$ form on a $(k-1)$-cube

Let ω be a k form on I^k

$\forall p \in I^k \quad \omega(p) \in \Lambda^k(\mathbb{R}^k_p)$

$\omega = f(x^1, \dots x^k)\, dx^1 \wedge dx^2 \wedge \dots \wedge dx^k$

Definition:

We define $\displaystyle\int_{I^k} \omega = \int_{I^k} f(x^1, \dots x^k)\, dx^1 \wedge \dots \wedge dx^k$

$$\overset{\text{def}}{=} \int_{[0,1]^k} f(x^1, \dots, x^k)\, dx^1 dx^2 \dots dx^k$$

Riemann intergral

$$= \underbrace{\int_0^1 \left(\int_0^1 \left(\int_0^1 \dots \left(\int_0^1 f(x^1, \dots x^k\, dx^1 \right) dx^2 \right) \dots \right.}_{k \text{ intergrals.}} \dots dx^k$$

any order.

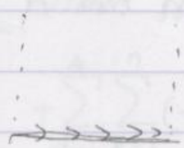
On $\mathbb{R}^k$, η be a $k-1$ form on $I^k_{(i,\alpha)}$

Basis of $k-1$ form in $\mathbb{R}^k$ is $dx^1 \wedge dx^2 \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge dx^k$

$j = 1, \dots k.$

Assume η is given by, $\eta = g(x^1, \dots x^k)\, dx^1 \wedge dx^2 \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge dx^k$

$$\int_{I^k(i,\alpha)} \eta = \begin{cases} \displaystyle\int_{[0,1]^{k-1}} g(x^1 \dots x^{j-1}, \alpha, x^{j+1}, \dots x^k)\, dx^1 \wedge dx^2 \wedge \dots \wedge \widehat{dx^j} \wedge \dots \wedge dx^k & i = j \\ 0 & i \neq j \end{cases}$$



$\displaystyle\int_{I^2_{(2,0)}} dy = 0$ $\qquad (x^1, x^2)$

$\displaystyle\int_{I^2_{(2,0)}} dx^2 \quad i = 2$

$y = 0$

missing dx^1.

If $\eta = \sum_{j=1}^{n} g_j(x^1,\dots,x^k)\,dx^1\wedge\dots\wedge\widehat{dx^j}\wedge\dots\wedge dx^k$

then $\int_{I^k_{(i,\alpha)}} \eta = \sum_{j=1}^{n} \int_{I^k(i,\alpha)} g_j(x^1,\dots,x^k)\,dx^1\wedge\dots\wedge\widehat{dx^j}\wedge\dots\wedge dx^k$

If ω is a 0-form then ω is a function $f(x^1,\dots x^k)$, 0-cube is point $\{0\}$

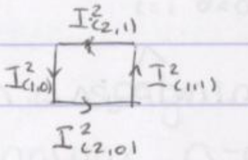
$$\int_{I^0} \omega = f(0,\dots 0)$$

If $c = \sum_{j=1}^{m} m_j c_j$ where c_j are all k-cubes (standard) then

$$\int_c \omega = \sum_{j=1}^{m} m_j \int_{c_j} \omega$$

If $c = \sum_{j=1}^{m} m_j c_j$ where c_j are $k-1$ cubes, then

$$\int_c \eta = \sum_{j=1}^{m} m_j \int_{c_j} \eta$$

eg. 

$$\int_{\partial I^2} \overset{1\text{-form}}{\eta} = \int_{I^2_{(2,0)}} \eta + \int_{I^2_{(1,1)}} \eta - \int_{I^2_{(2,1)}} \eta - \int_{I^2_{(1,0)}} \eta$$

Poincaré Lemma.

If A is star-shaped wrt 0 and ω is closed ~~&~~ l-form on A then ω is exact.

Proof: $\omega = \sum_{i_1<\dots<i_l} \omega_{i_1\dots i_l}\,dx^{i_1}\wedge\dots\wedge dx^{i_l}$

$$I(\underset{l\text{-form}}{\omega}) = \sum_{i_1<\dots<i_l} \sum_{\alpha=1}^{l} (-1)^{\alpha-1} \int_0^1 t^{l-1}\,\omega_{i_1\dots i_l}(tx)\,dt\; x^{i_\alpha}\,dx^{i_1}\wedge\dots\wedge\widehat{dx^{i_\alpha}}\wedge\dots\wedge dx^{i_l}$$

$I(0)=0 \quad I(\omega_1+\omega_2) = I(\omega_1) + I(\omega_2)$

If ω is closed $d\omega = 0$

$dI(\omega) + I(d\omega) = \omega$ ← unproved!

If ω is close $dI(\omega) + \underbrace{I(d\omega)}_{I(0)=0} = \omega$

$\Rightarrow dI(\omega) = \omega \Rightarrow \omega$ is exact

Because I is a linear operator it suffices to prove it for $\omega = f(x^1,\dots x^n)\, dx^{i_1}\wedge\dots\wedge dx^{i_l}$

$$d\omega = \sum_{\beta=1}^n D_\beta f(x^1\dots x^n)\, dx^\beta\wedge dx^{i_1}\wedge\dots\wedge dx^{i_l}$$

$$dI(\omega) = \sum_{\beta=1}^n\sum_{\alpha=1}^l (-1)^{\alpha-1}\int_0^1 t^{l-1} D_\beta\left(f(tx)x^{i_\alpha}\right)dt\, dx^\beta\wedge dx^{i_1}\wedge\dots\wedge\widehat{dx^{i_\alpha}}\wedge\dots\wedge dx^{i_l}$$

$$= \sum_{\beta=1}^n\sum_{\alpha=1}^l (-1)^{\alpha-1}\int_0^1 t^{l-1}\left(\delta_{(\beta,i_\alpha)} f(tx) + x^{i_\alpha}(D_\beta f)(tx)\cdot t\right)dt\, dx^\beta\wedge dx^{i_1}\wedge\dots\wedge\widehat{dx^{i_\alpha}}\wedge\dots$$

$$= \sum_{\alpha=1}^l (-1)^{\alpha-1}\int_0^1 t^{l-1} f(tx)\,dt\, dx^{i_\alpha}\wedge dx^{i_1}\wedge\dots\wedge\widehat{dx^{i_\alpha}}\wedge\dots dx^{i_k}$$
$$+ \sum_{\beta=1}^n\sum_{\alpha=1}^l (-1)^{\alpha-1}\int_0^1 t^l x^{i_\alpha}(D_\beta f)(tx)\,dt\, dx^\beta\wedge dx^{i_1}\wedge\dots\wedge\widehat{dx^{i_\alpha}}\wedge\dots\wedge dx^{i_l}$$

$$= \cancel{\sum_{\alpha=1}^l}\cancel{(-1)^{\alpha-1}(-1)^{\alpha-1}}\, l\int_0^1 t^{l-1} f(tx)\,dt\, dx^{i_1}\wedge\dots\wedge dx^{i_\alpha}\wedge\dots\wedge dx^{i_l}$$

not dependent on α

$$+ \sum_{\beta=1}^n\sum_{\alpha=1}^l (-1)^{\alpha-1}\int_0^1 t^l (D_\beta f)(tx)\,dt\, x^{i_\alpha} dx^\beta\wedge\dots\wedge\widehat{dx^{i_\alpha}}\wedge\dots\wedge dx^{i_l}$$

$$I(d\omega) = \sum_{j\in\{\beta,i_1,\dots,i_l\}}\sum_{\beta=1}^n (-1)^{j-1}\int_0^1 t^l D_\beta f(tx)\,dt\, x^j\, dx^\beta\wedge\dots\wedge\widehat{dx^j}\wedge\dots\wedge dx^{i_l}$$

$$= \sum_{\beta=1}^n \cancel{(-1)^{\beta-1}}\int_0^1 t^l (D_\beta f)(tx)\,dt\, x^\beta dx^{i_1}\wedge\dots\wedge dx^{i_l}$$
$$+ \cancel{\sum_{\alpha=1}^l\sum_{\beta=1}^n (-1)^\alpha\int_0^1 t^l (D_\beta f)(tx)\,dt\, x^{i_\alpha} dx^\beta\wedge\dots\wedge\widehat{dx^{i_\alpha}}\wedge\dots\wedge dx^{i_l}}$$

cancels in next step with above

$$dI(\omega) + I(d\omega) = \int_0^1 l t^{l-1} f(tx)\,dt\, dx^{i_1}\wedge\dots\wedge dx^{i_l}$$
$$+ \sum_{\beta=1}^n\int_0^1 t^l (D_\beta f)(tx)\,dt\, x^\beta dx^{i_1}\wedge\dots\wedge dx^{i_l}$$

$$= \left(\int_0^1\left[l t^{l-1} f(tx) + \sum_{\beta=1}^n x^\beta t^l (D_\beta f)(tx)\right]dt\right) dx^{i_1}\wedge\dots\wedge dx^{i_l}$$

$$= \int_0^1 \frac{d}{dt}\left(t^i f(tx)\right) dt \; dx^{i_1} \wedge \dots \wedge dx^{i_i}$$

$$= t^i f(tx)\Big|_{t=0}^{t=1} dx^{i_1} \wedge \dots \wedge dx^{i_i}$$

$$= f(1 \cdot x) dx^{i_1} \wedge \dots \wedge dx^{i_i} - 0 \quad = \omega.$$

Stokes Theorem

$$\int_{\partial c} \omega = \int_c d\omega$$

ω $k-1$ form

$d\omega$ k form

c k-singular chain

∂c $(k-1)$ singular chain.

We defined $\int_c \omega$ if ω is in $\cancel{\mathbb{R}^k}$ k-form (in $\mathbb{R}^k$)

c standard k-cube $c = I^k = [0,1]^k$

$\int_{I^k_{(i,\alpha)}} \eta$ $\quad \eta$ $k-1$ form.

Proof of Stokes Theorem on $\mathbb{R}^k$ for ω $k-1$ form, $c = I^k$ standard k-cube:

$$\int_{\partial I^k} \omega = \int_{I^k} d\omega$$

We know $\int_c \eta$ is linear η ie $\int_c \lambda\eta_1 + \eta_2 = \lambda \int_c \eta + \int_c \eta$

Therefore it suffices to prove it for

$\omega = f(x^1, x^2, \dots x^k) dx^1 \wedge dx^2 \wedge \dots \wedge \hat{dx^j} \wedge \dots \wedge dx^k$

$$d\omega = \sum_{\beta=1}^{k} D_\beta f \, dx^\beta \wedge dx^1 \wedge \dots \wedge \widehat{dx^j} \dots \wedge dx^k$$

$$= D_j f \, dx^j \wedge dx^1 \wedge \dots \wedge \widehat{dx^j} \wedge \dots \wedge dx^k$$
$$= (-1)^{j-1} D_j f \, dx^1 \wedge dx^2 \wedge \dots \wedge dx^j \wedge \dots \wedge dx^k$$

$$\int_{I^k} d\omega = (-1)^{j-1} \int_{I^k} D_j f \, dx^1 \wedge dx^2 \wedge \dots \wedge dx^k$$

$$\overset{\text{def}}{=} (-1)^{j-1} \int_{[0,1]^k} D_j f \, dx^1 dx^2 \dots dx^k$$

$$= (-1)^{j-1} \int_0^1 \dots \int_0^1 \left(\int_0^1 D_j f \, dx^j \right) dx^1 dx^2 \dots \widehat{dx^j} \dots dx^k$$
$$= (-1)^{j-1} \int_0^1 \dots \int_0^1 f(x^1, x^2 \dots x^k) \Big|_{x^j=0}^{x^j=1} dx^1 \dots \widehat{dx^j} \dots dx^k$$
$$= (-1)^{j-1} \int_0^1 \dots \int_0^1 f(x^1, x^2 \dots 1, x^{j+1} \dots x^k) \, dx^1 \dots \widehat{dx^j} \dots dx^k$$
$$- (-1)^{j-1} \int_0^1 \dots \int_0^1 f(x^1, x^2, \dots 0, x^{j+1}, \dots x^k) dx^1 \dots \widehat{dx_j} \dots dx^k.$$

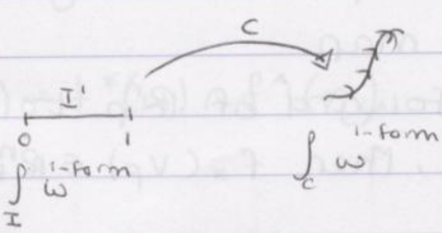
$$\int_{\partial I^k} \omega \overset{\text{def}}{=} \sum_{i=1}^{k} \sum_{\alpha=0,1} (-1)^{i+\alpha} \int_{I^k_{(i,\alpha)}} \overset{k-1}{\omega}$$

$$\overset{\text{def}}{=} \sum_{\alpha=0,1} (-1)^{j+\alpha} \int_{[0,1]^{k-1}} f(x^1, x^2 \dots \alpha, x^{j+1} \dots x^k) \, dx^1 dx^2 \dots \widehat{dx^j} \dots dx^k$$

only $i=j$ terms remain.

$$= (-1)^{j+1} \int_{[0,1]^{k-1}} f(x^1, x^2 \dots, 1, x^{j+1} \dots x^k) dx^1 \dots \widehat{dx^j} \dots dx^k$$
$$+ (-1)^j \int_{[0,1]^{k-1}} f(x^1, x^2 \dots 0, x^{j+1} \dots x^k) \, dx^1 \dots \widehat{dx^j} \dots dx^k$$

Consider



We will define $c^*(\omega)$ pull back.

Definition:

If ω is k-form on A containing singular k-cube c then $(c : I^k \to A)$

$$\int_c \omega = \int_{I^k} c^*(\omega)$$

How to define the pullback of a k-form by c?

Remembe if

$$\begin{tikzcd} V \arrow[r, "S"] \arrow[dr, "T\circ S"'] & {} \arrow[d, "T"] \\ & \mathbb{R} \end{tikzcd}$$

S linear, V, W VS, T linear function then $S^*(T) = T \circ S$

Pullback of tensors

$T \in \mathcal{J}^k(W)$ then $S^*(T) \in \mathcal{J}^k(V)$

$S^*(T)(v_1 \dots v_k) = T(S(v_1), \dots, S(v_k))$ $\quad v_i$'s $\in V$.

Let ω be a k-~~tensor~~form on $\mathbb{R}^m$ and let $f : \mathbb{R}^n \to \mathbb{R}^m$

To define $\underbrace{f^*(\omega)}_{k\text{-form on } \mathbb{R}^n}(p) \in \Lambda^k(\mathbb{R}^n_p)$ $\forall p \in \mathbb{R}^n$

$$f^*(\omega)(p)(v_1, v_2 \dots v_k) = \underbrace{\omega(f(p))}_{\in\, \Lambda^k(\mathbb{R}^m_{f(p)})}(Df(v_1), Df(v_2) \dots Df(v_k))$$

$v_i \in \mathbb{R}^n_p$

Definition:

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable ~~pe~~

$p \in \mathbb{R}^n$ $\quad Df(p) : \mathbb{R}^n \to \mathbb{R}^m$ linear map

It helps us define the push-foward of $\mathbb{R}^n_p$ to $\mathbb{R}^m_{f(p)}$

If $v_p \in \mathbb{R}^n_p$ $\quad v_p = (p, v)$ $\quad v \in \mathbb{R}^n$, then $f_*(v_p) \in \mathbb{R}^m_{f(p)}$ defined by $f_*(v_p) = (f(p), Df(p)(v))$

$f_* : \mathbb{R}^n_p \to \mathbb{R}^m_{f(p)}$ is linear

If ~~ma~~ $\underset{(p,v)}{v_p}, \underset{(p,w)}{w_p} \in \mathbb{R}^n_p$ $\quad \lambda \in \mathbb{R}$

$$
\begin{aligned}
f_*(\lambda v_p + w_p) &= f_*(\lambda(p, v) + (p, w)) \\
&= f_*((p, \lambda v + w)) \\
&\overset{\text{def}}{=} (f(p), Df(p)(\lambda v + w)) \\
&\overset{Df \text{ linear}}{=} (f(p), \lambda Df(p)(v) + Df(p)(w)) \\
&\overset{\text{def}}{=} \lambda (f(p), Df(p)(v)) + (f(p), Df(p)(w)) \\
&\overset{\text{def}}{=} \lambda f_*(v_p) + f_*(w_p)
\end{aligned}
$$

Definition:

If $T \in \mathcal{J}^k(\mathbb{R}^m_{f(p)})$ then $f^*(T)$ will be defined by

$$f^*(T)(v_1, v_2 \ldots v_k) = T(f_*(v_1), f_*(v_2) \ldots f_*(v_k)) \quad v_i \in \mathbb{R}^n_p.$$

If ω is a k-form on $\mathbb{R}^m$ then $f^*(\omega)$ is a k-form on $\mathbb{R}^n$ defined by $(\forall p \in \mathbb{R}^n)$.

$$f^*(\omega)(p)(v_1, v_2 \ldots v_k) = \omega(f(p))(f_*(v_1), f_*(v_2) \ldots f_*(v_k))$$

$v_i \in \mathbb{R}^n_p.$

Proposition: $f: \mathbb{R}^n \to \mathbb{R}^m$ differentiable.

i) $f^*(dx^i) = \sum_{j=1}^{n} D_j f^i \, dx^j$

ii) $f^*(\lambda\omega_1 + \omega_2) = \lambda f^*(\omega_1) + f^*(\omega_2)$

iii) $f^*(g\omega) = (g \circ f) f^*(\omega)$

iv) $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$

Example

ω 1-form in $\mathbb{R}^3$

$\omega = P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz$

$f: [0,1] \to \mathbb{R}^3$ parametrizes a curve in $\mathbb{R}^3$.

$f^*(\omega)$ 1-form on $[0,1]$

$t \in [0,1]$ $f^*(\omega)$ has to be ? dt.

Let v_t be a tangent vectors on $\mathbb{R}^1_t$ $v_t = (t, v)$

$f^*(\omega)(t) \in (v_t) = \omega(f(t))(f_*(v_t))$

$\Lambda^1(\mathbb{R}^1_t)$

$= (Pdx + Qdy + Rdz)(f(t))(f_*(v_t))$

$\overset{iii}{=} P(f(t))\,dx(f(t))(f_*(v_t)) + Q(f(t))\,dy(f(t))(f_*(v_t)) + R(f(t))\,dz(f(t))(f_*(v_t)).$

$f = (f^1, f^2, f^3)$ $= P(f(t))Df^1(t)(v) + Q(f(t))Df^2(t)(v) + R(f(t))Df^3(t)(v).$

$= Df^*(\omega) = (P\circ f)\dfrac{df^1}{dt}dt + (Q\circ f)\dfrac{df^2}{dt}dt + (R\circ f)\dfrac{df^3}{dt}dt$

$f^*(\omega) = f^*(Pdx + Qdy + Rdz)$

$\overset{iii}{=} (P\circ f)f^*(dx) + (Q\circ f)f^*(dy) + (R\circ f)f^*(dz)$

$\overset{i}{=} (P\circ f)Df^1 dt + (Q\circ f)Df^2 dt + (R\circ f)Df^3 dt$

Proof of proposition:

i) $f^*(dx^i) = \sum_{j=1}^{n} D_j f^i dx^j$

1-form on $\mathbb{R}^m$ 1-form on $\mathbb{R}^m$

Take $p \in \mathbb{R}^m$, $f^*(dx^i)(p) \in \Lambda^1(\mathbb{R}^m_p)$

$f^*(dx^i)(p)(v_p) = dx^i(f(p))(f_*(v_p))$ $v_p = (p, v) \in \mathbb{R}^m_p$

$= dx^i(f(p))(f(p), Df(p)(v)).$

dx^i picks up i-component of vector $= (f(p), Df(p)(v))^i$

$= \sum_{j=1}^{n} D_j f^i(p) v^j$

$f: \mathbb{R}^n \to \mathbb{R}^m$

$Df(p)(v) = f'(p)v$

i^{th} row $\to$
$$= \begin{pmatrix} D_1f^1 & D_2f^1 & \cdots & D_nf^1 \\ D_1f^2 & D_2f^2 & \cdots & D_nf^2 \\ \vdots & & & \vdots \\ D_1f^m & D_2f^m & \cdots & D_nf^m \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Compose $\left(\sum_{j=1}^{n} D_j f^i dx^j\right)(p)(v_p) = \sum_{j=1}^{n} D_j f^i(p)\, dx^j(p)(v_p)$

$= \sum_{j=1}^{n} D_j f^i(p) v^j$

iii) $f^*(g\omega) = (g\circ f) f^*(\omega)$ $\quad f: \mathbb{R}^{m} \to \mathbb{R}^{n}$ $g: \mathbb{R}^n \to \mathbb{R}$.

$p \in \mathbb{R}^{*}$ $\quad v_1 \dots v_k \in \mathbb{R}^n_p$

$$f^*(g\omega)(p)(v_1,\dots v_k) = (g\omega)(f(p))(f_*(v_1)\dots f_*(v_k))$$
$$= g(f(p))\,\omega(f(p))(f_*(v_1)\dots f_*(v_k))$$

compare $(g\circ f) f^*(\omega)(p)(v_1\dots v_k) = (g\circ f)(p)\,\omega(f(p))(f_*(v_1)\dots f_*(v_k))$

Definition:

If $c: I^k \to A$ is a singular k cube in A and ω is a k-form on A then

$$\int_c \omega \overset{\text{def.}}{=} \int_{I^k} c^*(\omega)$$

Example

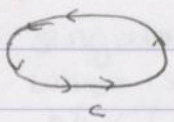
ω 1-form on $\mathbb{R}^2$, $\omega = x\,dy$

$c: [0,1] \to \mathbb{R}^2$

$c(t) = (a\cos(2\pi t), b\sin(2\pi t))$ $\quad a, b > 0.$

$$\int_c x\,dy = \int_{[0,1]} c^*(x\,dy)$$
$$= \int_0^1 (x\circ c)(t)\,\frac{dc^2}{dt}\,dt$$
$$= \int_0^1 a\cos(2\pi t)\, b2\pi\cos(2\pi t)\,dt$$
$$= \int_0^1 2\pi ab\cos^2(2\pi t)\,dt$$
$$= ab\pi\int_0^1 1+\cos(4\pi t)\,dt = \pi ab.$$ — are of ellipse

Stokes Theorem, $\int_c \omega = \int_{\tilde{c}} d\omega = \int_{\tilde{c}} d(x\,dy) = \int_{\tilde{c}} dx\,dy.$ area of region parametes by c.



call $\tilde{c}$ the inside of the ellipse 2-cube

$\tilde{c}(u,t) = (au\cos(2\pi t), bu\sin(2\pi t))$

$t\in[0,1],\ u\in[0,1]$ $\quad \partial\tilde{c} = c.$

Definition:

If c is a singular k-chain, ie $c = \sum_{j=1}^{m} m_j c_j$ $m_j \in \mathbb{Z}$, c_j singular k-~~chain~~ cubes.

$$\int_c \omega = \sum_{j=1}^{m} m_j \int_{c_j} \omega = \sum_{j=1}^{m} m_j \int_{I^k} c_j^*(\omega)$$

Stokes Theorem for singular k-chains in $\mathbb{R}^k$

ω $k-1$ form on $\mathbb{R}^k$

$d\omega$ k form on $\mathbb{R}^k$

c k-singular chain

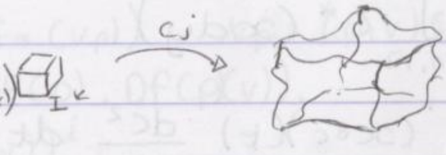
∂c $k-1$ singular chain

Then,

$$\int_{\partial c} \omega = \int_c d\omega$$

Note: $\partial c_j = c_j(\partial I^k)$

$$= \sum_{i=1}^{k} \sum_{\alpha=0,1} (-1)^{i+\alpha} c_j(I^k_{(i,\alpha)})$$



Proof: $\int_{\partial c} \omega = \sum_{j=1}^{m} m_j \int_{\partial(c_j)} \omega$

$$= \sum_{j=1}^{m} \sum_{i=1}^{k} \sum_{\alpha=0,1} (-1)^{i+\alpha} m_j \int_{c_j(I^k_{(i,\alpha)})} \omega$$

$$\overset{\text{by def}}{=} \sum_{j=1}^{m} \sum_{i=1}^{k} \sum_{\alpha=0,1} \left(m_j (-1)^{i+\alpha} \int_{I^k_{(i,\alpha)}} c_j^*(\omega)\right)$$

Now compute $\int_c d\omega = \sum_{j=1}^{m} m_j \int_{c_j} d\omega$ ← singular k cube

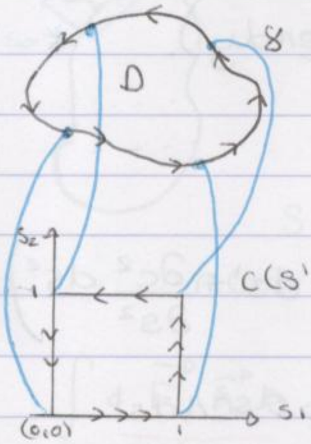
$$\overset{\text{def}}{=} \sum_{j=1}^{m} m_j \int_{I^k} c_j^*(d\omega) = \sum m_j \int_I d(c_j^* \omega)$$

since $d(c_j^*\omega) = c_j^*(d\omega)$

$$= \sum_{j=1}^{m} m_j \int_{\partial I^k} c_j^*(\omega) \quad \text{by Stokes Thm for standard k-}$$

$$= \sum_{j=1}^{m} m_j \sum_{i=1}^{k} \sum_{\alpha=0,1} (-1)^{i+\alpha} \int_{I^k_{(i,\alpha)}} c_j^*(\omega)$$

(Classical) Stokes Theorem in $\mathbb{R}^2$



$$\int_\gamma P(x,y)\,dx + Q(x,y)\,dy$$

$$= \iint_D \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) dx\,dy.$$

$c(s^1,s^2) = (c^1(s^1,s^2), c^2(s^1,s^2))$

$\gamma : [0,1] \to \mathbb{R}^2 \quad \gamma(t) = (\gamma^1(t), \gamma^2(t))$

$$\partial c \overset{\text{by def}}{=} c(\partial I^2) = \gamma$$

$$\int_\gamma P\,dx + Q\,dy = \int_{c(\partial I^2)} P\,dx + Q\,dy \overset{\text{by def}}{=} \int_{\partial I^2} c^*(P\,dx + Q\,dy)$$

$$= \int_{\partial I^2} P(c^1(s^1,s^2), c^2(s^1,s^2)) \underbrace{c^*(dx)}_{\gamma^*(dx)} + Q(c^1(s^1,s^2), c^2(s^1,s^2)) \underbrace{c^*(dy)}_{\gamma^*(dy)}$$

$$= \int_{\partial I^2} P \frac{d\gamma^1}{dt}\,dt + Q \frac{d\gamma^2}{dt}\,dt$$

$$= \int_{\partial I^2} \left[P(\gamma^1(t),\gamma^2(t)) \frac{d\gamma^1}{dt} + Q(\gamma^1(t),\gamma^2(t)) \frac{d\gamma^2}{dt}\right] dt$$

$$\int_{\partial c} \underbrace{P\,dx + Q\,dy}_{\omega} \overset{\text{Stokes in } \mathbb{R}^2}{=} \int_c \underbrace{d(P\,dx + Q\,dy)}_{d\omega}$$

γ = ∂c singular 1-cube which is boundary of $c(I^2)$.

$$\int_{\partial c} \omega = \int_c d\omega$$

$$\int_c d(Pdx + Qdy) = \int_c P_x \cancel{dx\wedge dx} + P_y\, dy\wedge dx + Q_x\, dx\wedge dy + Q_y\, \cancel{dy\wedge dy}$$

$$= \int_c -\frac{\partial P}{\partial y}\, dx\wedge dy + \frac{\partial Q}{\partial x}\, dx\wedge dy$$

$$= \int_c \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) dx\wedge dy.$$

$$\overset{\text{by}}{\underset{\text{det}}{=}} \int_{I^2} c^*\left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) dx\wedge dy$$

$$= \int_{I^2}\left(-\frac{\partial P}{\partial y}(c^1, c^2) + \frac{\partial Q}{\partial y}(c^1, c^2)\right) c^*(dx\wedge dy). \quad *$$

What is $c^*(dx\wedge dy) = c^*(dx)\wedge c^*(dy)$

$$= \left(\frac{\partial c^1}{\partial s^1} ds^1 + \frac{\partial c^1}{\partial s^2} ds^2\right)\wedge\left(\frac{\partial c^2}{\partial s^1} ds^1 + \frac{\partial c^2}{\partial s^2} ds^2\right)$$

$$= \frac{\partial c^1}{\partial s^1}\frac{\partial c^2}{\partial s^2} ds^1\wedge ds^2 + \frac{\partial c^1}{\partial s^2}\frac{\partial c^2}{\partial s^1} ds^2\wedge ds^1$$

$$= \left(\frac{\partial c^1}{\partial s^1}\frac{\partial c^2}{\partial s^2} - \frac{\partial c^1}{\partial s^2}\frac{\partial c^2}{\partial s^1}\right) ds^1\wedge ds^2$$

$$= \det c'(s^1, s^2)\, ds^1\wedge ds^2$$

$$* = \int_{I^2}\left(-\frac{\partial P}{\partial y}(c^1, c^2) + \frac{\partial Q}{\partial x}(c^1, c^2)\right)\det c'(s^1, s^2)\, ds^1 \cancel{\wedge} ds^2$$

ordinary double intergral.

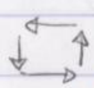
Recall change of variables formula for n-dim intergrals

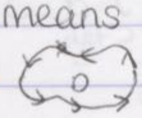
$A \subseteq \mathbb{R}^n \quad g: A \to \mathbb{R}^n$ injective, differentiable

$\det g'(x) \neq 0 \quad \forall x \in A$

If $f: g(A) \to \mathbb{R}$ is intergrable, $\displaystyle\int_{g(A)} f = \int_A (f\circ g)|\det g'|$

multiple intergrals

Assuming  means going around it can be shown that

$\det c'(s^1, s^2) > 0$ 

$$\int_{I^2}\left(-\frac{\partial P}{\partial y}(c^1,c^2)+\frac{\partial Q}{\partial x}(c^1,c^2)\right)\det c'(s^1,s^2)\,ds^1ds^2$$

$$=\iint_D -\frac{\partial P}{\partial y}(x,y)+\frac{\partial Q}{\partial x}(x,y)\,dx\,dy.$$

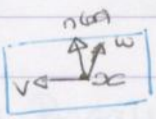
Gauß or Divergence Theorem.

Solid T in $\mathbb{R}^3$ with boundary surface S
vector field $F=(F^1,F^2,F^3)$

S_x tangent plane to the solid at point $x\in S$
$n(x)$ is outwards unit normal vector.

$$\int_S \underbrace{\langle \vec{F},\vec{n}\rangle}_{\text{scalar product}}\,dA = \iiint_T (\operatorname{div}\vec{F})\,dx\,dy\,dz$$

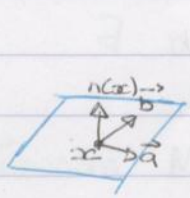
has dim 2, tangent plane at x.
$\dim \Lambda^2(S_x)=1.$ $\left(=\binom{2}{2}\right).$



$v,w\in S_x$

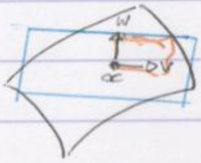
$$\omega(v,w)=(v\times w)\cdot n$$
$$=\langle v\times w, n\rangle \quad \text{scalar triple product.}$$

$$\omega(v,w)=\begin{vmatrix} v^1 & v^2 & v^3\\ w^1 & w^2 & w^3\\ n^1 & n^2 & n^3\end{vmatrix}$$



choose $\vec{a},\vec{b}\in S_x$ st $\vec{a},\vec{b},\vec{n}$ are orthonormal, right handed system
$\omega(\vec{a},\vec{b})=1\neq 0.$

Notation: Call $\omega(v,w) = dA(v,w)$, $\omega = dA$ where $\omega(\vec{a},\vec{b}) = 1$.



Theorem:

$$dA = n^1\, dy\wedge dz + n^2\, dz\wedge dx + n^3\, dx\wedge dy$$

Proof:

$$dA(v,w) = \begin{vmatrix} n^1 & n^2 & n^3 \\ v^1 & v^2 & v^3 \\ w^1 & w^2 & w^3 \end{vmatrix} = n^1(v^2w^3 - w^2v^3) + n^2(-v^1w^2 + v^3w^1) + n^3(v^1w^2 - w^1v^2)$$

$$\begin{aligned}(dy\wedge dz)(v,w) &= (dy\otimes dz - dz\otimes dy)(v,w) \\ &= dy(v)dz(w) - dz(v)dy(w) \\ &= v^2w^3 - v^3w^2\end{aligned}$$

$$(dz\wedge dx)(v,w) = v^3w^1 - v^1w^3$$

$$(dx\wedge dy)(v,w) = v^1w^2 - \cancel{w}^2w^1$$

Theorem:

$n^1 dA = dy\wedge dz$

$n^2 dA = dz\wedge dx$

$n^3 dA = dx\wedge dy$

← Proof: we know that $(dy\wedge dz)(v,w) = v^2w^3 - v^3w^2$ where $v,w \in S_x$

$dA(v,w) = \langle v\times w, n\rangle$

Since v and w are perpendicular to $\vec{n}$

$v\times w = \lambda n$, $\lambda\in\mathbb{R}$

$n^1 dA(v,w) = n^1\langle \lambda n, n\rangle = n^1\cdot\lambda$ since $|n| = 1$.

$$\langle v\times w, i\rangle = \langle \lambda n, i\rangle$$

$$= \lambda n^1$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v^1 & v^2 & v^3 \\ w^1 & w^2 & w^3 \end{vmatrix}\cdot i = v^2w^3 - v^3w^2 = (dy\wedge dz)(v,w)$$

Proof of Gauss Theorem:

Given $\vec{F} = (F^1, F^2, F^3) = F^1\underline{i} + F^2\underline{j} + F^3\underline{k}$.

$$\operatorname{div}(\vec{F}) = \frac{\partial F^1}{\partial x} + \frac{\partial F^2}{\partial y} + \frac{\partial F^3}{\partial z}.$$

To F we assign the 2-form $\eta = F^1 dy\wedge dz + F^2 dz\wedge dx + F^3 dx\wedge dy$ (2-form)

Calculate $d\eta = \frac{\partial F^1}{\partial x} dx\wedge dy\wedge dz + \frac{\partial F^2}{\partial y} dy\wedge dz\wedge dx + \frac{\partial F^3}{\partial z} dz\wedge dx\wedge dy$

(3-form) $= \left(\frac{\partial F^1}{\partial x} + \frac{\partial F^2}{\partial y} + \frac{\partial F^3}{\partial z}\right) dx\wedge dy\wedge dz$

$$\int_T \partial\eta = \int_T (\operatorname{div} F)\, dx\wedge dy\wedge dz = \iiint_T \operatorname{div} F \cdot dx\,dy\,dz$$

singular 3-cube

change of variables for I^3 to T.

$$\int_T \partial\eta \overset{\text{stokes}}{=} \int_{\partial T} \eta = \int_S F^1 dy\wedge dz + F^2 dz\wedge dx + F^3 dx\wedge dy$$

$$= \int_S F^1 n^1 dA + F^2 n^2 dA + F^3 n^3 dA$$

$$= \int_S (F^1 n^1 + F^2 n^2 + F^3 n^3)\, dA = \int_S \vec{F}\cdot\vec{n}\, dA$$

Recall

Definition:

M is a K-dim manifold in $\mathbb{R}^n$ if for all $x \in M$

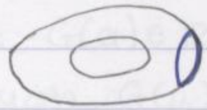
(M) $\exists$ U open set in $\mathbb{R}^n$, V open set in $\mathbb{R}^n$, $x \in U$

$\exists$ differmorphism $h: U \to V$ st $h(U\cap M) = V \cap \{y \in \mathbb{R}^n, y^{k+1} = \dots = y^n = 0\}$

Theorem: M is a k-dim field in $\mathbb{R}^n$ iff $\forall x \in M$, condition C holds

(C) $\exists$ W open in $\mathbb{R}^k$, $\exists$ U open in $\mathbb{R}^n$ $x \in U$

$\exists f : W \to U$ st f is injective, $\operatorname{rank} f'(y) = k \ \forall y \in W$
$f(W) = U \cap M$, $f^{-1} : U \cap M \to W$ continuous.



← consider

Definition:

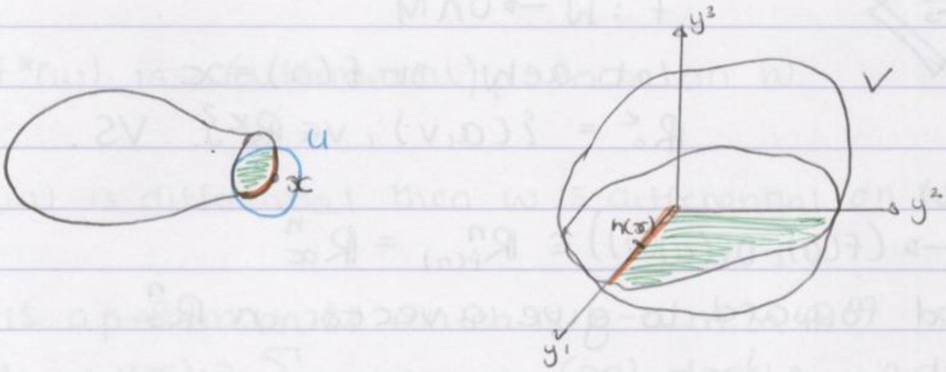
A subset M of $\mathbb{R}^n$ is a k-dim manifold with boundary if $\forall x \in M$ either (M) holds or (exclusive) (M') holds

(M') $\exists$ open set U of $\mathbb{R}^n$, $x \in U$

$\exists$ ~~$\forall$~~ V open set in $\mathbb{R}^n$, h differmorphism, $h: U \to V$

$h(U \cap M) = V \cap \{y,\ y^k \geq 0,\ y^{k+1} = \dots = y^n = 0$

$h^k(x) = 0$.



The boundary ~~of~~ ∂M of M is defined to be the set of points x where condition M' holds.

$f'(y)$ has rank k for all $y \in W$

Nullity + rank theorem

$k = \dim \mathbb{R}^k = \dim \ker(Df(y)) + \text{rank}(Df(y))$

$Df(y) : \mathbb{R}^k \to \mathbb{R}^n$ linear

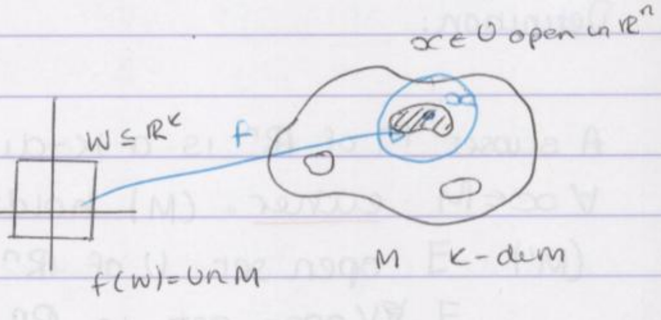
Since $f : \mathbb{R}^k \to \mathbb{R}^n$

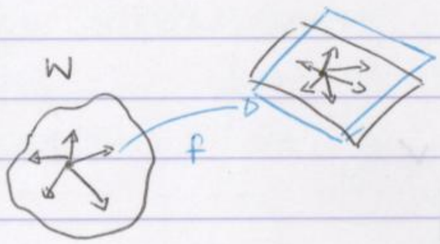
$k = \text{nullity} + k$

$\Rightarrow \text{nullity} = 0$

$\ker(Df(y)) = \{\vec{0}\}$

$Df(y)$ is injective





$f : W \to U \cap M$

Let $a \in W$ st $f(a) = x$

$\mathbb{R}^k_a = \{(a, v),\ v \in \mathbb{R}^k\}$. VS.

$\forall v \in \mathbb{R}^k \quad (a, v) \to (f(a), Df(a)(v)) \in \mathbb{R}^n_{f(a)} = \mathbb{R}^n_x$

(a, v) is pushed foward to give a vector in $\mathbb{R}^n_x$

$v_a \in = (a, v)$

$f_*(v_a) = (x, Df(a)(v)) \in \mathbb{R}^n_x$

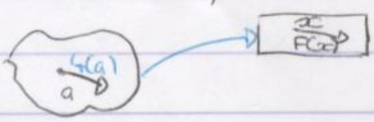
Definition:

The tangent space of M, at x is defined to be $M_x = f_*(\mathbb{R}^k_a)$ ($\dim M_x = k$) (given $x = f(a)$, f chart).

Definition:

A vector field on M is a function F on M st $\forall x \in M$, $F(x) \in M_x$

Let $x = f(a)$, $f : W \to U$ $f(W) = U \cap M$



Let $G(a) \in \mathbb{R}^k_a$ st $f_a(G(a)) = F(f(a)) = F(x)$

Such $G(a)$ is unique since $f_x : \mathbb{R}^k_a \to M_x$ is bijective

Definition:

F a vector field on M is called continuous (or differentiable) if $\forall x \in M$ the vector field G on W is continuous (or differentiable)

Definition:

ω is a (differential) form on M, if $\forall x \in M$ $\omega(x) \in \Lambda^p(M_x)$

Then $f^*(\omega)$ is a (differential) p-form on W

If $f^*(\omega)$ is differential then ω is differential on $W \subseteq \mathbb{R}^k$

If ω is a p-form on M which is k-dim in $\mathbb{R}^n$

$x \in M$, $\omega(x) = \sum_{i_1 < i_2 < \dots < i_p} \omega_{i_1, i_2 \dots i_p}(x) \, dx^{i_1} \wedge \dots \wedge dx^{i_p}$.

ω is continuous if $f^*(\omega)$ is continuous on W

ω is differentiable if $f^*(\omega)$ is differentiable on W

We have difficulty with $D_j(\omega_{i_1 \dots i_p}(x))$ since $\omega_{i_1 \dots i_p}(x)$ is not defined on an open set $U \ni x$.

Theorem:

Given a differential form p-form on M which is k-dim in $\mathbb{R}^n$, there exists a unique differential $(p+1)$-form $d\omega$ on M st $\forall x \in M$ and $f : W \to U \cap M$ chart, $d(f^*(\omega)) = f^*(d\omega)$.

Proof: ~~dw~~ $dw(x) \in \Lambda^{p+1}(M_x)$, $v_i \in M_x$

$dw(x)(v_1, v_2, \ldots v_{p+1})$

Since f_* is bijection $f_* : \mathbb{R}^k_a \to M_x$

$\exists$ unique vectors $w_1, w_2, \ldots w_{p+1} \in \mathbb{R}^k_a$ st $f_*(w_i) = v_i$

$$dw(x)(v_1 \ldots v_{p+1}) = \underbrace{df^*(w)(a)}_{\in \Lambda^{p+1}(\mathbb{R}^k_a)}(w_1, w_2 \ldots w_{p+1})$$

Aim: To understand Stokes' Theorem for M, k-dim manifold in $\mathbb{R}^n$ with boundary ∂M.

$$\int_{\partial M} \omega = \int_M d\omega$$

where ω is a $k-1$ differential ~~man~~ form on M, $d\omega$ is a k-form on M.

Orientation on Vector spaces.

$\mathcal{F} = \langle v_1, v_2, \ldots v_n \rangle$ ordered bases

$\mathcal{B} = \langle w_1, w_2, \ldots w_n \rangle$

We say $\mathcal{F}$ & $\mathcal{B}$ define the same orientation if $\det [Id]^{\mathcal{B}}_{\mathcal{F}} > 0$

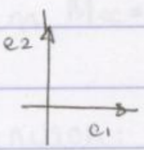
(oppisite orientation $\det [Id]^{\mathcal{B}}_{\mathcal{F}} < 0$).

$[Id]^{\mathcal{B}}_{\mathcal{F}} = ([Id]^{\mathcal{F}}_{\mathcal{B}})^{-1}$

$\mathcal{F} \sim \mathcal{B}$ iff they define same orientation

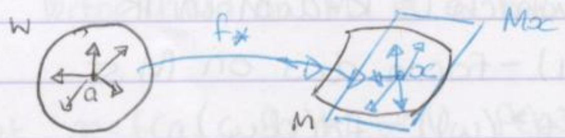
This is an equivalence relation

Standard orientation in $\mathbb{R}^n$ $\mathcal{F} = \langle e_1, e_2, \ldots, e_n \rangle$



$\langle e_1, e_2 \rangle$ has oppisite orientation to $\langle e_2, e_1 \rangle$

The standard orientation is denoted $\mu = [e_1, e_2, \ldots, e_n]$



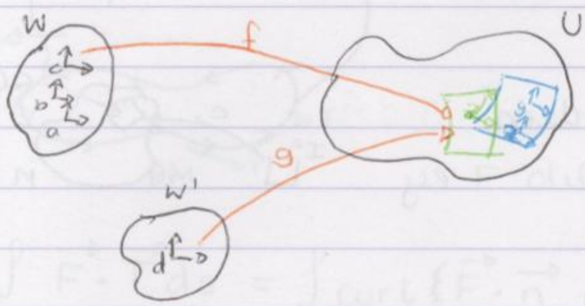
$f(a) = x$

On $\mathbb{R}^k_a$ we have the standard basis $\{(e_1)_a, (e_2)_a, \ldots (e_k)_a\}$.

Base for M_x $\{f_*((e_1)_a), f_*((e_2)_a) \ldots f_*((e_k)_a)\}$.

$M_x = [f_*((e_1)_a), f_*((e_2)_a), \dots f_*((e_k)_a)]$

If $b \in W$ then ~~M_x~~ $\mu_{f(b)} = [f_*((e_1)_b), \dots f_*((e_k)_b)]$



$Z = f(c)$

$Z = g(d)$

We assign two orientations at Z

$[f_*((e_1)_c), f_*((e_2)_c) \dots f_*((e_k)_c)]$

$= [g_*((e_1)_c), g_*((e_2)_c), \dots g_*((e_k)_c)]$

If the two orientations are equal

ie $\det [Id] > 0$ on these two bases, then we say f and g define consistent orientation at point Z.

Hopefully, this is true on $f(W) \cap g(W')$ then we call the two orientations consistent

If there exists consistent orientations on all of M, we say M is orientable and the manifold is ~~orientable~~ oriented once we fix orientation.

if S is a surface in $\mathbb{R}^3$ which is orientable let $M_x = [v_1, v_2]$

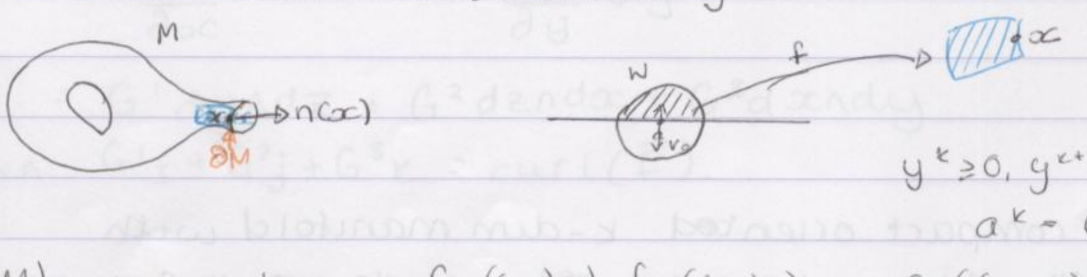
$x \in S$ (2-manifold)

Draw the line perpendicular to S_x at x

Pick a unit vector $n(x)$ st $[n(x), v_1, v_2]$ is the standard orientation in $\mathbb{R}^3$.

Then $n(x)$ is the (outer) unit normal.

M is k-dim manifold with boundary in $\mathbb{R}^n$



$f(a) = x$

$y^k \geq 0,\ y^{k+1} = y^{k+1} = \dots = 0$

$a^k = 0$

$(\partial M)_x$ has a basis, $f_*((e_1)_a), f_*((e_2)_a), \dots, f_*((e_{k-1})_a)$

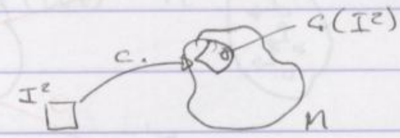
Then let $v_0 \in \mathbb{R}^k_a$ st $f_*(v_0)$ perpendicular at β

then $|f_x(v_0)| = 1$, then $n(x) = f_x(v_0)$

Intergrals.

Let c be a singular p-cube on M k-dim.

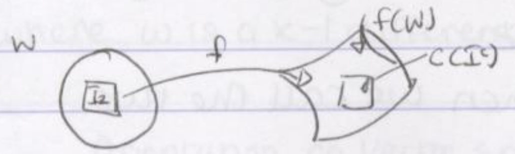
$c: I^k \longrightarrow M$

eg 

Let ω be a p-form on M

We define $\int_c \omega = \int_{I^k} c^*(\omega)$

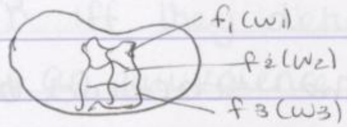
If c is a k-cube in M, k-dim manifold and $I^k \subseteq W$,
$f: W \to U \cap M$ is the chart and $c = f$, $\forall x \in I^k$



f is preserving orientation then we say c is orientation preserving singular k-cube on M
If ω is k-form on M with $\omega(y) = 0$ $\forall y \notin c(I^2)$, then we define

$$\int_M \omega = \int_c \omega$$

Consider



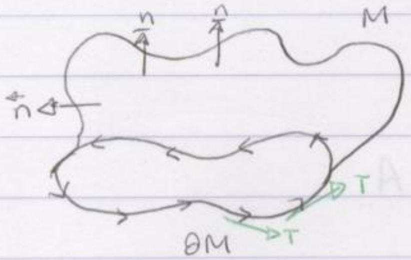
$$\int_{f_1(w_1)} \omega + \int_{f_2(w_2)} \omega + \dots = \int_M \omega.$$

Use partitions of unity to define $\underset{k\text{-form}}{\int_M \omega}$, $\underset{k-1 \text{ form}}{\int_{\partial M} \eta}$

Theorem:

Let M be a compact oriented k-dim manifold with boundary ∂M and ω be a differentiable $k-1$ form on M. Then $\int_{\partial M} \omega = \int_M d\omega$

Classical Stokes Theorem.



$\mathbb{R}^3$

M is a 2-dim manifold, oriented, with boundary. F differential vectorfield on M.

$$\int_{\partial M} \vec{F}\cdot\vec{T}\,ds = \int_M \operatorname{curl}\{\vec{F}\cdot\vec{n}\,dA$$

Let M be a compact oriented 2-dim manifold with boundarys ∂M in $\mathbb{R}^3$.
Let T be a vector field on ∂M st $ds(T)=1$ where ds is the length element of ∂M.
Let $\vec{F}$ be a diff. vector field on an open set containing M.
$\vec{n}$ be the outer normal on M. Then,

$$\int_{\partial M} \vec{F}\cdot\vec{T}\,ds = \int_M \operatorname{curl}\vec{F}\cdot\vec{n}\,dA$$

Proof: If $F=(F^1,F^2,F^3) = F^1\underline{i}+F^2\underline{j}+F^3\underline{k}$ we define 1-form $\omega = F^1dx + F^2dy + F^3dz$ then we calculate.

$$d\omega = \frac{\partial F^1}{\partial y}dy\wedge dx + \frac{\partial F^1}{\partial z}dz\wedge dx + \frac{\partial F^2}{\partial x}dx\wedge dy + \frac{\partial F^2}{\partial z}dz\wedge dy + \frac{\partial F^3}{\partial x}dx\wedge dz + \frac{\partial F^3}{\partial y}dy\wedge dz$$

$$= G^1dy\wedge dz + G^2dz\wedge dx + G^3dx\wedge dy$$

Then $G^1\underline{i}+G^2\underline{j}+G^3\underline{k} = \operatorname{curl}(\vec{F})$.

Last lecture $dy\wedge dz = n^1dA$, $dz\wedge dx = n^2dA$, $dx\wedge dy = n^3dA$

$$\int_M G^1 dy\wedge dz + G^2 dz\wedge dx + G^3 dx\wedge dy$$

$$= \int_M (G^1n^1 + G^2n^2 + G^3n^3)\,dA$$

$$= \int_M \vec{G}\cdot\vec{n}\,dA = \int_M \operatorname{curl}\vec{F}\cdot n\,dA.$$

According to general stokes theorem,

$$\int_{\partial M} \omega = \int_M d\omega = \int_M (\operatorname{curl}\vec{F})\cdot\vec{n}\,dA$$

Since $ds(T) = 1$, we can prove as in previous lecture that, $dx = T^1 ds$, $dy = T^2 ds$, $dz = T^3 dz$

$$\int_{\partial M} \omega = \int_{\partial M} F^1 dx + F^2 dy + F^3 dz$$

$$= \int_{\partial M} F^1 T^1 ds + F^2 T^2 ds + F^3 T^3 ds$$

$$= \int_{\partial M} \vec{F}\cdot\vec{T}\,ds.$$